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Approximate Nash Equilibria Algorithms for Shapley Network Design Games

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Abstract: We consider a weighted Shapley network design game, where selfish players choose paths in a network to minimize their cost. The cost function of each edge in the network is affine linear with respect to the sum of the weights of the players choosing the edge. We first show the existence of an α-approximate pure Nash equilibrium by constructing a potential function and establish an upper bound $O(\log_2(W))$ on α, where W is the sum of the weights of all players. Furthermore, we assume that the coefficients of the cost function (affine linear function) of the edge are all ϕ-smooth random variables on $[0, 1]$. In this case, we show that ϵ-best response dynamics can compute the $(1+\epsilon)\alpha$-approximate pure Nash equilibrium (ϵ is a positive constant close to 0) in polynomial time by proving that the expected number of iterations is polynomial in $\frac{1}{\epsilon}$, ϕ, the number of players and the number of edges in the network.

Key words: Artificial intelligence; Approximate Nash equilibrium; Affine linear cost function; Best response dynamics; Potential function

1 Introduction

As networks such as the Internet continue to expand, the number of self-interested players using them also grows, each optimizing an individual objective function. It is natural to study the behavior of these non-cooperative players in the game, as has been done in previous work, including[1–6]. Furthermore, game theory has multiple applications in computer science, especially concerning deep learning models[7–9]. For example, Generative Adversarial Network (GAN) is a deep learning architecture that has roots in game theory[10, 11]. Practically, the training of GANs can be modeled as a two-player zero-sum game played by the generator and the discriminator.

Specifically, a game can be defined on a directed network and each player needs to choose a path from a source vertex to a sink vertex. Each edge in the network has a cost and the players using the edge share the cost equally or according to their weights. All players want to reduce their personal cost and choose the path simultaneously, forming a profile of the game. The stable profiles are called *Nash equilibria*, where no player has an incentive to change strategy unilaterally. Rosenthal[12] first studies the equilibrium of the game on a network and proved the existence of a *pure Nash equilibrium* (PNE) in the game. Then Anshelevich et al.[13] name the game *network design with Shapley cost sharing*, also known as *Shapley network design game*

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or *fair cost sharing game*, since it uses the cost sharing mechanism.

Furthermore, the study of PNE gives rise to a number of new issues. In particular, since the overall system performance may be worse when decisions are made by selfish players compared to centralized authority decision-making, we need an indicator to measure the extent to which the global performance of the system has deteriorated due to selfish decision-making. Papadimitriou[14] first proposes using the term *price of anarchy* to refer to the deterioration of system performance. Moreover, in some types of games there may be multiple PNEs, but the price of anarchy studies only the degradation of the worst PNE. Then Anshelevich et al.[13] introduce the term *price of stability* to refer to the ratio of the cost of the best Nash equilibrium to the optimal network cost. Moreover, Anshelevich et al.[13] prove that a Shapley network design game with n players always admits a PNE and the price of stability is at most $H(n) = \sum_{i=1}^{n} \frac{1}{i}$. This is achieved by using the *potential function* method introduced by Monderer and Shapley[15].

However, Anshelevich et al.[13] prove only a few results when players are *weighted*, including proving the existence of PNE in the two-player weighted Shapley network design game. Chen and Roughgarden[16] present a three-player weighted Shapley network design game that does not admit a PNE, motivating the subsequent study of *α-approximate pure Nash equilibrium* (α-APNE). In our model, α-APNE is a profile such that no player can unilaterally change strategy to reduce the cost by at least a factor of α. Then Chen and Roughgarden[16] show that the existence of $O(\log w_{\max})$-APNE in any weighted Shapley network design game and prove that the price of stability with respect to the $O(\log w_{\max})$-APNE is $O(\log W)$, where $w_{\max}$ and W are the maximum weight and the sum of weights of all players respectively. Moreover, they construct a network without $o(\frac{\log w_{\max}}{\log\log w_{\max}})$-APNE.

In addition, computing a Nash equilibrium is essential when the game is given from a computational point of view. Apart from proposing the Shapley network design game, Anshelevich et al.[13] show that *best response dynamics* may take exponential time to converge to a PNE. Specifically, best response dynamics is an algorithm for finding a PNE (an α-APNE) by picking a player who has a strategy that can reduce personal cost (by more than α times) and updating the strategy as long as the profile is not a PNE (an α-APNE). Furthermore, Syrgkanis[17] shows that computing a PNE in the unweighted Shapley network design game is Polynomial Local Search (PLS)-complete by the reduction from MAX CUT. Therefore, it is necessary to make further assumptions about the game in order to ensure that there exists an algorithm that can find a PNE or an α-APNE in polynomial time. Giannakopoulos[18] assumes the constant cost c_e of the edge as ϕ-smooth independent random variables on $[0, 1]$, which means its probability density function is upper bounded by ϕ. And Giannakopoulos[18] proves that best response dynamics can compute the α-APNE in polynomial time under this assumption.

In real-world scenarios, the cost of each edge in the network is not always a constant, but rather may vary based on the number of players (or the sum of the weights of players) using the edge. However, to the best of our knowledge, there are currently no positive results available for the weighted Shapley network design game where the cost of each edge is not a constant. Therefore, we extend the cost of each edge to a linear function to the total weights of the players that choose the edge. Under this assumption, we prove the existence of α-APNE, present an algorithm to compute α-APNE and analyze its time complexity.

1.1 Related Works

Epstein et al.[5] prove the existence of *strong Nash equilibria* of the unweighted Shapley network design game, where strong Nash equilibria are profiles from which no group of players can decrease the cost of each of its members. Moreover, Epstein et al.[5] show that the *strong price of anarchy* of a fair cost sharing game with n players is at most $H(n) = \sum_{i=1}^{n} \frac{1}{i}$, namely $H(n)$ is the n-th harmonic number.

For the undirected Shapley network design game with n players, Anshelevich et al.[13] show that the upper bound of the price of stability is $H(n) = \sum_{i=1}^{n} \frac{1}{i}$. Furthermore, Christodoulou et al.[19] present a lower bound of the price of stability of $42/23 \approx 1.8261$, which is improved to $348/155 \approx 2.245$ by Bilò et al.[4]. Fiat et al.[20] further restrict the network such that there is a common source vertex and a player in every vertex, and prove that the price of stability of the game is $O(\log\log n)$. This further restricted game is called as *broadcast network design game*, and the price of stability of the game is improved to constant by Bilò et al.[21].

When the game does not apply the cost sharing mechanism, Fabrikant et al.[1] consider the game in which each player bears the whole cost of each chosen edge and name it the *network congestion game*. They prove that computing a Nash equilibrium is PLS-complete. Moreover, when all players have the same source vertex and sink vertex, namely the game is *symmetric*, Fabrikant et al.[1] show that there is a polynomial algorithm for finding a PNE. Furthermore, Ackermann et al.[22] show that the PLS-completeness holds in the undirected network congestion game.

1.2 Our contributions

Our model is the first to extend the cost function to an affine linear function in the weighted Shapley network design game proposed by Anshelevich et al.[13], which is a constant c_e for each edge e in the network. In our model, the cost function of each edge in the network $c_e(w_e) = a_e w_e + b_e$, where $a_e, b_e > 0$ refer only to the edge e and w_e is the total weight of the players choosing a path containing the edge e. By constructing an α-approximate potential function $\Phi(P) = \sum_{e \in E} c_e(w_e) \log_2(1 + w_e)$, we prove the existence of α-APNE, where $\alpha = 2\log_2(1 + W + w_{\max}) = O(\log_2(W))$, where W and $w_{\max}$ are the sum and the maximum of the weights of all players respectively.

To compute the α-APNE, we adopt ϵ-best response dynamics, which can yield a $(1+\epsilon)\alpha$-APNE. Under the assumption that $\{a_e, b_e\}_{e \in E}$ are ϕ-smooth random variables, which means that the probability density function of a_e, b_e is upper bounded by a constant ϕ, we prove that the expected number of iterations is polynomial in $\frac{1}{\epsilon}, \phi$, the number of players n and the number of edges m in the network. In particular, we find out that the expected number of iterations can be bounded by $O(\frac{1}{\epsilon} W^3 \phi n^2 m \cdot \ln(W \phi n m))$. Since we can adopt the shortest-path search at each iteration step to check for the existence of $(1+\epsilon)\alpha$-improving deviations, the ϵ-best response dynamics can compute the $(1+\epsilon)\alpha$-APNE in polynomial time.

The remainder of the paper is organized as follows. Section 2 defines the model and several concepts as preliminaries. Section 3 proves the existence of α-APNE. Section 4 proposes an algorithm to compute the α-APNE in polynomial time. Conclusions and possible directions in the future are presented in Section 5.

2 Model and Preliminaries

We represent a *weighted Shapley network design game* Γ by tuple $(\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ with components defined below.

- **P1:** $\mathcal{G} = (V, E)$ is a directed graph, where V is the vertex set of $\mathcal{G}$ and E is the edge set of $\mathcal{G}$. Here we further define $m = |E|$, which is the number of edges in graph $\mathcal{G}$.
- **P2:** $\mathcal{N}$ is a collection of n different players $(n \geq 2)$. Moreover, each player $i \in \mathcal{N}$ is identified with a source-sink vertex pair (s_i, t_i).
- **P3:** $\mathcal{P} = (\mathcal{P}_i)_{i \in \mathcal{N}}$, where $\mathcal{P}_i$ denote the set of simple $s_i - t_i$ paths. Furthermore, each player i chooses a path $P_i \in \mathcal{P}_i$ from its source vertex to its sink vertex.
- **P4:** $\mathcal{C} = (c_e)_{e \in E}$, where $c_e(\cdot)$ is the non-negative cost function of the edge e.
- **P5:** $\mathcal{W} = (w_i)_{i \in \mathcal{N}}$, where each component $w_i > 0$ is a positive weight of player $i \in \mathcal{N}$. When $w_i = w_j$ for any $i, j \in \mathcal{N}$, then Γ is called unweighted. Otherwise, Γ is called weighted.

All players are selfish, and each of them chooses a path independently. Then it results in a profile $P = (P_i)_{i \in \mathcal{N}}$, where P_i denotes the simple $s_i - t_i$ path used by player $i \in \mathcal{N}$. For a given profile and an edge $e \in E$, we define $U_e(P) := \{i | i \in \mathcal{N}, e \in P_i\}$ as the "user" of edge e. Moreover, we define $w_e = \sum_{i \in U_e(P)} w_i$ as the total weight of the players choosing a path containing edge e. Furthermore, for a given profile and a player i, the cost share c_e^i of an edge $e \in E$ is $c_e(P) \cdot w_i / w_e$. Then the cost to player i in a profile is the sum of its cost shares: $c_i(P) = \sum_{e \in P_i} c_e^i$. In addition, the cost $c(P)$ of a profile P is defined as $\sum_{i \in \mathcal{N}} c_i(P)$, which can be also written as $\sum_{e \in \cup_{i \in \mathcal{N}} P_i} c_e(P)$.

To state our discussion precisely, for each player $i \in \mathcal{N}$, we denote by P_{-i} a sub-profile $(P_{i'})_{i' \in \mathcal{N} \setminus \{i\}}$ of strategies used by "opponents" $i' \in \mathcal{N} \setminus \{i\}$ of player i, and use $P = (P_i, P_{-i})$ when we want to explicitly show the strategy P_i used by player i.

Definition 1 (Pure Nash equilibrium) Given an instance $(\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game Γ. We call a profile $P = (P_i)_{i \in \mathcal{N}}$ a pure Nash equilibrium (PNE) if, for each i, P_i minimizes $c_i(P)$ over all paths in $\mathcal{P}_i$ while keeping P_j fixed for $j \neq i$. That is:

$$c_i(P) = c_i(P_i, P_{-i}) \leq c_i(P_i', P_{-i}) \tag{1}$$

for any $i \in \mathcal{N}$, any $P_i' \in \mathcal{P}_i$.

In fact, Inequality (1) states that P_i is a *best-response* to P_{-i} for player i. Therefore, there is no incentive for any player $i \in \mathcal{N}$ to unilaterally change strategy when the profile is already a PNE.

To prove that a PNE exists, a common approach is to prove that there is a *potential function* in the game. Here we give the definition of the potential function.

Definition 2 (Exact potential function) Given an instance $(\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game Γ. We call a real-valued function $\Phi : \prod_{i\in\mathcal{N}} \mathcal{P}_i \to [0, \infty)$ an (exact) potential function if

$$\Phi(P_i', P_{-i}) - \Phi(P_i, P_{-i}) = c_i(P_i', P_{-i}) - c_i(P_i, P_{-i}) \tag{2}$$

for any $i \in \mathcal{N}$, any $P_i, P_i' \in \mathcal{P}_i$, any $P_{-i} \in \prod_{j\in\mathcal{N}\setminus\{i\}} \mathcal{P}_j$.

If there is a potential function in a finite game, since the potential function quantifies the cost reduction of a unilateral change of strategy exactly, then the global minimums of the potential function are the PNEs of the game.

However, in some cases, a game may not admit a PNE. In such cases, a relaxed notion of an *α-approximate pure Nash equilibrium* must be considered.

Definition 3 (α-Approximate pure Nash equilibrium) Given an arbitrary instance $(\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game Γ and an arbitrary constant $\alpha \geq 1$. We call a profile $P = (P_i)_{i\in\mathcal{N}}$ an α-approximate pure Nash equilibrium (α-APNE) if

$$c_i(P) = c_i(P_i, P_{-i}) \leq \alpha \cdot c_i(P_i', P_{-i}) \tag{3}$$

for any $i \in \mathcal{N}$, any $P_i' \in \mathcal{P}_i$.

In fact, Inequality (3) states that a unilateral change of strategy reduces the cost by at most $\frac{c_i(P_i,P_{-i})-c_i(P_i',P_{-i})}{c_i(P_i,P_{-i})} \leq 1 - \frac{1}{\alpha}$ in an α-APNE. Moreover, we call a deviation *α-improving* if the deviation decreases the cost incurred by the player by at least an α multiplicative factor. Therefore, α-APNE is a profile from which no α-improving deviations exist. We further point out that the smaller the constant α is, the closer the profile P would approximate a PNE.

Correspondingly, we can prove the existence of α-APNE by finding the approximate potential function for the game.

Definition 4 (α-Approximate potential function) Given an instance $(\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game Γ. For any player $i \in \mathcal{N}$, $P_i, P_i' \in \mathcal{P}_i$ satisfies $P_i \to P_i'$ is an α-improving deviation. We call a real-valued function $\Phi : \prod_{i\in\mathcal{N}} \mathcal{P}_i \to [0, \infty)$ an α-approximate potential function if

$$\Phi(P_i', P_{-i}) - \Phi(P_i, P_{-i}) \leq \alpha c_i(P_i', P_{-i}) - c_i(P_i, P_{-i}) \tag{4}$$

for any $P_{-i} \in \prod_{j\in\mathcal{N}\setminus\{i\}} \mathcal{P}_j$.

If there is a α-approximate potential function in a finite game, since the potential function strictly decreases in every α-improving deviation, then there will be no α-improving deviation after a finite number of steps, which means the game admits an α-APNE.

Without loss of generality, we assume $\min_{i\in\mathcal{N}} w_i = 1$ in the following discussion, and we define $w_{\max} = \max_{i\in\mathcal{N}} w_i$ and $W = \sum_{i\in\mathcal{N}} w_i$ to simplify our discussion.

3 Existence of α-APNE

Anshelevich et al.[13] prove that the unweighted Shapley network design game admits a PNE, and point out that the weighted Shapley network design game does not admit a PNE. Since our model is an extension of the weighted Shapley network design game, we have to turn our attention to α-APNE instead of PNE.

Theorem 1 Given an instance $(G, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game Γ. If the cost function is affine linear, namely $c_e(w_e) = a_e w_e + b_e$, where $a_e, b_e > 0$ are only related to the edge e and w_e is the total weight of the players that choose a path containing edge e, then the game admits an α-APNE, where $\alpha = O(\log_2(W))$.

Proof For a weighted Shapley network design game Γ, we define a function Φ as follows:

$$\Phi(P) = \sum_{e\in E} c_e(w_e) \log_2(1 + w_e).$$

Now we consider an α-improving deviation of player i from the profile $P = (P_i, P_{-i})$ to the profile $P' = (P_i', P_{-i})$. Without loss of generality, we can assume that P_i and P_i' are disjoint. Otherwise, we can replace P_i and P_i' with $P_i \backslash P_i'$ and $P_i' \backslash P_i$ in the following discussion. By the definition of α-improving, we have

$$\alpha \cdot \sum_{e\in P_i'} c_e(w_e + w_i) \cdot \frac{w_i}{w_e + w_i} \leq \sum_{e\in P_i} c_e(w_e) \cdot \frac{w_i}{w_e}.$$

Here, w_e denotes the total weight on edge e before the deviation and we let $\alpha = 2\log_2(1 + W + w_{\max}) = O(\log_2(W))$. Since $c_e(w_e) = a_e w_e + b_e$, we have

$$\alpha \cdot \sum_{e\in P_i'} [a_e(w_e + w_i) + b_e] \cdot \frac{w_i}{w_e + w_i} \leq \sum_{e\in P_i} (a_e w_e + b_e) \cdot \frac{w_i}{w_e}. \tag{5}$$

Then we can derive the following:

$$
\begin{aligned}
\Delta\Phi = & \sum_{e\in P_i'} (a_e w_e + b_e)\cdot \\
& \qquad [\log_2(1+w_e+w_i) - \log_2(1+w_e)] \\
& - \sum_{e\in P_i} (a_e w_e + b_e)\cdot \\
& \qquad [\log_2(1+w_e) - \log_2(1+w_e-w_i)] \\
& + \sum_{e\in P_i'} a_e w_i \log_2(1+w_e+w_i) \\
& - \sum_{e\in P_i} a_e w_i \log_2(1+w_e-w_i) \\
< & \sum_{e\in P_i'} (a_e w_e + b_e) \log_2[e(1+w_i)] \frac{w_i}{w_e+w_i} \\
& - \sum_{e\in P_i} (a_e w_e + b_e) \frac{w_i}{w_e} \\
& + \sum_{e\in P_i'} a_e w_i \log_2(1+w_e+w_i) \\
= & \sum_{e\in P_i'} a_e w_i \{\log_2(1+w_e+w_i) \\
& \qquad + \log_2[e(1+w_i)] \frac{w_e}{w_e+w_i}\} \\
& + \sum_{e\in P_i'} b_e \log_2[e(1+w_i)] \frac{w_i}{w_e+w_i} \\
& - \sum_{e\in P_i} (a_e w_e + b_e) \frac{w_i}{w_e} \\
< & \ \alpha \sum_{e\in P_i'} [a_e(w_e+w_i) + b_e] \cdot \frac{w_i}{w_e+w_i} \\
& - \sum_{e\in P_i} (a_e w_e + b_e) \cdot \frac{w_i}{w_e} \\
= & \ \alpha c_i(P') - c_i(P).
\end{aligned}
$$

For the first inequality, we take the logarithm of $e(1+w_i) \geq (1+\frac{w_i}{w_e+1})^{\frac{w_e+1}{w_i}}(1+\frac{w_i}{w_e+1}) \geq (1+\frac{w_i}{w_e+1})^{\frac{w_e+w_i}{w_i}}$ and apply $\log_2(\frac{1+w_e}{1+w_e-w_i}) > 1 \geq \frac{w_i}{w_e}$ when $2w_i \geq 1+w_e$ or take the logarithm of $(1+\frac{w_i}{1+w_e-w_i})^{\frac{w_e}{w_i}} \geq (1+\frac{w_i}{1+w_e-w_i})^{\frac{1+w_e-w_i}{w_i}} \geq 2$ when $2w_i < 1+w_e$. Moreover, we use $\alpha \geq \max\{\log_2[e(1+w_i)], \log_2(1+w_e+w_i)+\log_2[e(1+w_i)]\frac{w_e}{w_e+w_i}\}$ for the last inequality. Therefore, we have proved

$$\Phi(P') - \Phi(P) < \alpha c_i(P') - c_i(P), \tag{6}$$

which suggests that α-improving deviations strictly decrease the function Φ. Since the number of different profiles are finite, the theorem can be directly derived. ■

4 Compute the $(1+\epsilon)\alpha$-APNE

Initially we propose an algorithm to find the $(1+\epsilon)\alpha$-APNE of the game Γ as follows, where ϵ is a positive constant close to 0.

Algorithm 1 ϵ-approximate best response dynamics(ϵ-ABRD)

Input: A weighted Shapley network design game $\Gamma = (\mathcal{G}, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$; a strategy profile $P \in \mathcal{P}$; a constant $\alpha > 0$; a constant $\epsilon > 0$

Output: A $(1+\epsilon)\alpha$-APNE of game Γ

(1) **While** P is not an $(1+\epsilon)\alpha$-APNE **do**

(2) Choose $i \in \mathcal{N}$, $P_i' \in \mathcal{P}_i$ such that
$$(1+\epsilon)\alpha c_i(P_i', P_{-i}) < c_i(P) = c_i(P_i, P_{-i})$$

(3) $P \leftarrow (P_i', P_{-i})$

(4) **end while**

(5) **return** P

Specifically, the first and the second lines of the Algorithm 1 are to find an $(1+\epsilon)\alpha$-improving deviation for player i. For a given instance of game Γ, we may not be able to search exhaustively in the strategy space $\mathcal{P}_i$ since this would take exponential time. Instead, we can do a shortest-path search for player i, which can efficiently find the best response for player i in polynomial time. This way, we only need to check whether the best response is an $(1+\epsilon)\alpha$-improving deviation, which can also be computed in polynomial time. Therefore, in order to show that the Algorithm 1 runs in polynomial time, we only need to prove that the number of iterations is polynomial in the given parameters. Before we begin to prove this, we first make an assumption about the parameters and propose a lemma as preparation.

Definition 5 Let X be a random variable, we say X is a ϕ-smooth random variable if and only if X is defined on $[0,1]$ and the probability density function of X is bounded above by ϕ ($\phi \geq 1$).

In our model, it is natural to assume $\{a_e, b_e\}_{e\in E}$ are ϕ-smooth random variables. For ϕ-smooth random variable, we can derive the lemma below.

Lemma 1 Let $X_1, X_2, ..., X_n; Y_1, Y_2, ..., Y_n (n \geq 2)$ be ϕ-smooth independent random variables on [0,1]. Let $g(x) = \min\{\frac{\alpha}{x}\ln\frac{\alpha}{x}, n^\lambda\}$. Then for any $\lambda \in \mathbb{N}$ and

$\alpha \geq 1$, we have

$$\mathbb{E}[g(\min_{i\in[n]}\{X_i + Y_i\})] \leq \alpha\phi(n^2+1)\cdot\ln(\alpha\phi n). \quad (7)$$

Proof For two random variables S, T, we define $S \preceq T$ if and only if $F_T(x) \leq F_S(x)$ for any $x \in \mathbb{R}$, where $F_S(x), F_T(x)$ are the *cumulative distribution function* of S, T respectively. To prove the lemma, we define two sets of independent uniformly distributed random variables $S_1, S_2, ..., S_n; T_1, T_2, ..., T_n$ on $[0, \frac{1}{\phi}]$. In addition, we denote $[n] = \{1, 2, ...n\}$ to simplify our discussion.

Claim For any $i \in [n]$, we have

$$\min_{i\in[n]} S_i \preceq \min_{i\in[n]} X_i, \quad \min_{i\in[n]} T_i \preceq \min_{i\in[n]} Y_i.$$

To prove the claim, we firstly prove $F_{X_i}(t) \leq F_{S_i}(t)$ for any $t \in \mathbb{R}$, then we can derive $F_{Y_i}(t) \leq F_{T_i}(t)$ similarly.

Since S_i is defined on $[0, \frac{1}{\phi}]$, we have $F_{S_i}(t) = 1$ for any $t \geq \frac{1}{\phi}$, then we have $F_{X_i}(t) \leq F_{S_i}(t)$ definitely. When $0 \leq t \leq \frac{1}{\phi}$, since X_i is a ϕ-smooth random variable for any $i \in [n]$, its probability density function is upper-bounded by ϕ. Then we have

$$F_{X_i}(t) \leq \int_0^t \phi dx = F_{S_i}(t)$$

for all $0 \leq t \leq \frac{1}{\phi}$. Therefore, we have $F_{X_i}(t) \leq F_{S_i}(t)$ and $F_{Y_i}(t) \leq F_{T_i}(t)$. Furthermore, since $X_1, X_2, ..., X_n$ are independent random variables, we have

$$\begin{aligned} F_{\min_{i\in[n]} X_i}(t) &= 1 - \prod_{i=1}^{n}(1 - F_{X_i}(t)) \\ &\leq 1 - \prod_{i=1}^{n}(1 - F_{S_i}(t)) \\ &= F_{\min_{i\in[n]} S_i}(t), \end{aligned}$$

which completes the proof of the claim.

Moreover, we can find that $g(x) = \min\{\frac{\alpha}{x}\ln\frac{\alpha}{x}, n^\lambda\}$ is nonincreasing with respect to x. From the Proposition 9.1.2 in[23], we have

$$\mathbb{E}[g(\min_{i\in[n]}\{X_i + Y_i\})] \leq \mathbb{E}[g(\min_{i\in[n]}\{S_i + T_i\})]. \quad (8)$$

We define $W = \frac{1}{\min_{i\in[n]}\{S_i+T_i\}}$, then $W \in [\frac{\phi}{2}, \infty)$. Since S_i, T_i are independent uniformly distributed random variables on $[0, \frac{1}{\phi}]$, we have

$$F_W(w) = \prod_{i=1}^{n} Prob[S_i + T_i \geq \frac{1}{w}]$$
$$= \begin{cases} (1 - \dfrac{\phi^2}{2w^2})^n, & w \in [\phi, \infty) \\ (2 - \dfrac{2\phi}{w} + \dfrac{\phi^2}{2w^2})^n, & w \in [\dfrac{\phi}{2}, \phi) \end{cases}. \quad (9)$$

The last equation is due to Lemma 2 where we let $A \leftarrow \frac{1}{\phi}, t \leftarrow \frac{1}{w}$.

For $i = 1, 2..., \lambda, \lambda+1$, we define Q_i as the random event that $W \in [\phi n^{i-1}, \phi n^i)$. We let Q_0 and $Q_{\lambda+2}$ denote the random event that $W \in [\frac{\phi}{2}, \phi)$ and $W \in [\phi n^{\lambda+1}, \infty)$. Then, we can derive the following upper bound on the expectation in Inequality (8) by:

$$\begin{aligned} &\mathbb{E}[g(\min_{i\in[n]}\{X_i + Y_i\})] \\ &\leq \mathbb{E}[\min\{\alpha\cdot W\ln(\alpha\cdot W), n^\lambda\}] \\ &\leq \sum_{i=0}^{\lambda+1} Prob[Q_i]\cdot\mathbb{E}[\alpha\cdot W\ln(\alpha\cdot W)|Q_i] \\ &\quad + Prob[Q_{\lambda+2}]\cdot n^\lambda. \end{aligned}$$

The first inequality is due to the Inequality (8) And the second is due to the law of total expectation. Furthermore, we can derive the following:

$$\begin{aligned} &\mathbb{E}[g(\min_{i\in[n]}\{X_i + Y_i\})] \\ &\leq \sum_{i=1}^{\lambda+1}[F_W(\phi n^i) - F_W(\phi n^{i-1})]\cdot\alpha\phi n^i\cdot\ln(\alpha\phi n^i) \\ &\quad + [F_W(\phi) - F_W(\frac{\phi}{2})]\alpha\phi\ln(\alpha\phi) \\ &\quad + [1 - F_W(\phi n^{\lambda+1})]n^\lambda \\ &\leq \alpha\phi\cdot\ln(\alpha\phi n)\sum_{i=1}^{\lambda+1}[F_W(\phi n^i) - F_W(\phi n^{i-1})]\cdot in^i \\ &\quad + [F_W(\phi) - F_W(\frac{\phi}{2})]\alpha\phi\ln(\alpha\phi) \\ &\quad + [1 - F_W(\phi n^{\lambda+1})]n^\lambda \\ &= \alpha\phi\cdot\ln(\alpha\phi n)\cdot \\ &\qquad \sum_{i=1}^{\lambda+1}[(1 - \frac{1}{2n^{2i}})^n - (1 - \frac{1}{2n^{2i-2}})^n]\cdot in^i \\ &\quad + (\frac{1}{2})^n\cdot\alpha\phi\cdot\ln(\alpha\phi) + [1 - (1 - \frac{1}{2n^{2\lambda+2}})^n]n^\lambda \\ &\leq \alpha\phi\cdot\ln(\alpha\phi n)\cdot \\ &\qquad \sum_{i=1}^{\lambda+1}[\frac{1}{1+\frac{n}{2n^{2i}}} - (1 - \frac{n}{2n^{2i-2}})]\cdot in^i \\ &\quad + (\frac{1}{2})^n\cdot\alpha\phi\cdot\ln(\alpha\phi) + [1 - (1 - \frac{n}{2n^{2\lambda+2}})]n^\lambda \\ &\leq \alpha\phi\cdot\ln(\alpha\phi n)\cdot\sum_{i=1}^{\lambda+1}\frac{i}{2n^{i-3}} + (\frac{1}{2})^n\cdot\alpha\phi\cdot\ln(\alpha\phi) \\ &\quad + \frac{1}{2n^{\lambda+1}} \\ &\leq \alpha\phi\cdot\ln(\alpha\phi n)\cdot(2n^2+1). \end{aligned}$$

The third inequality is due to Lemma 3. Therefore, we complete the proof of the lemma. ■

It is worth mentioning that there is no λ on the right side of Inequality (7). With the formula for changing base, the second term of the function g has almost no effect on the conclusion.

Lemma 2 Let X, Y be two independent uniformly distributed random variables on $[0, A]$, where $A > 0$ is a constant. For a random variable X, $F_X(t)$ denotes the cumulative function of X. Then we have:

$$F_{X+Y}(t) = \begin{cases} \dfrac{t^2}{2A^2}, & t \in [0, A) \\ \dfrac{2t}{A} - \dfrac{t^2}{2A^2} - 1, & t \in [A, 2A] \end{cases}.$$

Proof We assume $Z = X + Y$, then Z is defined on $[0, 2A]$. For a random variable X, let $f_X(t)$ be the probability density function of X. Then we have $f_X(t) = f_Y(t) = \frac{1}{A}$ for any $t \in [0, A]$. Since X, Y are independent random variables, we have

$$f_Z(t) = \int_0^\infty f_X(x) f_Y(t-x)dx. \tag{10}$$

When $t \in [0, A)$, from Equation (10) we have

$$f_Z(t) = \int_0^t \frac{1}{A^2}dx = \frac{t}{A^2}.$$

Similarly, we can derive the following from Equation 10 when $t \in [A, 2A]$:

$$f_Z(t) = \int_{t-A}^A \frac{1}{A^2}dx = \frac{2A-t}{A^2}.$$

Furthermore, we have

$$\begin{aligned} F_Z(t) &= \int_0^t f_Z(x)\,dx \\ &= \begin{cases} \frac{t^2}{2A^2}, & t \in [0, A) \\ \frac{2t}{A} - \frac{t^2}{2A^2} - 1, & t \in [A, 2A]. \end{cases} \end{aligned}$$

■

Lemma 3 Let $x \in (0, 1), n \in N$, then we have:

(1) $(1-x)^n \geq 1 - nx$;

(2) $(1-x)^n \leq \frac{1}{1+nx}$.

Proof We use mathematical induction to prove part 1. When $n = 1$, the left and right sides of the inequality are equal. If the inequality holds when $n = k$, then for $n = k + 1$, we have

$$\begin{aligned} (1-x)^{k+1} &= (1-x)^k(1-x) \\ &\geq (1-kx)(1-x) \\ &= 1-(k+1)x + kx^2 \\ &> 1-(k+1)x. \end{aligned}$$

Therefore, by mathematical induction we prove part 1.

For part 2, from binomial expansion we have $(1 + x)^n \geq 1 + nx$. Then we can derive the following:

$$\begin{aligned} 1 &> (1-x^2)^n \\ &= (1+x)^n(1-x)^n \\ &\geq (1+nx)(1-x)^n. \end{aligned}$$

Therefore, we prove $(1-x)^n \leq \frac{1}{1+nx}$ and complete the proof of the lemma. ■

Now we begin to show that ϵ-ABRD always find the $(1 + \epsilon)\alpha$-APNE in polynomial time by proving the expected number of iterations is polynomial in the given parameters.

Theorem 2 Given an instance $(G, \mathcal{N}, \mathcal{P}, \mathcal{C}, \mathcal{W})$ of the game and $c_e(w_e) = a_e w_e + b_e$, where $\{a_e, b_e\}_{e \in E}$ are ϕ-smooth random variables. Then, the expected number of iterations of ϵ-ABRD is at most $O(\frac{1}{\epsilon}W^3\phi n^2 m \cdot \ln(W\phi nm))$.

Proof We recall that the approximate potential function Φ of the game Γ is

$$\Phi(P) = \sum_{e \in E} c_e(w_e)\log_2(1 + w_e),$$

which can be upper-bounded by

$$\begin{aligned} \Phi(P) &\leq \sum_{e \in E} c_e(W)\log_2(1 + w_e) \\ &\leq m \max_{e \in E}\{a_e W + b_e\}\log_2(1 + W). \end{aligned} \tag{11}$$

To simplify our discussion, we let $c_{\max} = \max_{e \in E}\{a_e W + b_e\}$. Since $\min_{i \in \mathcal{N}} w_i = 1$ and each player uses at least one edge, then Φ can be lower bounded by

$$\begin{aligned} \Phi(P) &\geq \sum_{e \in E} \min_{e \in E}\{a_e + b_e\}\log_2(1 + w_e) \\ &\geq \min_{e \in E}\{a_e + b_e\}\log_2(1 + W). \end{aligned} \tag{12}$$

The second inequality is due to $\log_2(1+x) + \log_2(1+y) \geq \log_2(1+x+y)$ for any $x, y > 0$. To simplify our discussion, we let $c_{\min} = \min_{e \in E}\{a_e + b_e\}$.

Furthermore, according to Algorithm 1, we assume that we move T steps $P^0 \rightarrow P^1 \rightarrow \ldots \rightarrow P^T$ during the execution of our ϵ-ABRD and $P \rightarrow P'$ is one of them. Since $c_i(P) \geq \frac{c_{\min}}{W}$, we can derive the following from Inequality (6) and Inequality (11):

$$\begin{aligned}\Phi(P)-\Phi(P') &> c_i(P)-\alpha c_i(P')\\ &> \frac{\epsilon}{1+\epsilon}c_i(P)\\ &\geq \frac{\epsilon}{(1+\epsilon)W}c_{\min}\\ &\geq \frac{\epsilon c_{\min}\Phi(P)}{(1+\epsilon)W\log_2(1+W)mc_{\max}}.\end{aligned} \tag{13}$$

Now we have

$$\Phi(P') < (1-\epsilon')\Phi(P),$$

where $\epsilon' = \dfrac{\epsilon c_{\min}}{(1+\epsilon)W\log_2(1+W)mc_{\max}}$. (14)

Since Inequality (14) holds for any move during the execution of the ϵ-ABRD, we can derive the following from Inequality (11) and Inequality (12):

$$\begin{aligned}c_{\min}\log_2(1+W) &\leq \Phi(P^T)\\ &< (1-\epsilon')^T\Phi(P^0)\\ &\leq (1-\epsilon')^T mc_{\max}\log_2(1+W).\end{aligned}$$

Therefore, we take logarithms at both sides of the inequality and get

$$T\ln(1-\epsilon') > \ln\frac{c_{\min}}{mc_{\max}}. \tag{15}$$

Since $0<\epsilon'<1$, we can derive the following upper bound on the number of steps for our ϵ-ABRD from Inequality (15):

$$\begin{aligned}T &< -\frac{1}{\ln(1-\epsilon')}\ln\frac{mc_{\max}}{c_{\min}}\\ &\leq \frac{1}{\epsilon'}\ln\frac{mc_{\max}}{c_{\min}}\\ &= (1+\frac{1}{\epsilon})W\log_2(1+W)\frac{mc_{\max}}{c_{\min}}\ln\frac{mc_{\max}}{c_{\min}}.\end{aligned} \tag{16}$$

The second inequality is due to $\epsilon' \geq \ln(\frac{1}{1-\epsilon'})$ when $0<\epsilon'<1$.

On the other hand, we recall that the approximate potential function Φ is strictly decreasing at every step of the ϵ-ABRD, which means no profile appears more than once during the ϵ-ABRD. Therefore, the total number of iterations cannot be larger than the number of different profiles, which is up to $(2^n)^m = 2^{nm}$. Combining this with Inequality (16) we have

$$T \leq \min\{\frac{1}{\epsilon'}\ln\frac{mc_{\max}}{c_{\min}}, 2^{nm}\}. \tag{17}$$

Since $c_{\min} = \min_{e\in E}\{a_e+b_e\}$ and $\{a_e, b_e\}_{e\in E}$ are ϕ-smooth random variables on $[0,1]$, then we can derive the following from Lemma 1:

$$\begin{aligned}\mathbb{E}[T] &\leq (1+\frac{1}{\epsilon})W\log_2(1+W)\cdot\\ &\qquad \mathbb{E}[\min\{\frac{m(1+W)}{c_{\min}}\ln\frac{m(1+W)}{c_{\min}}, m^{nm}\}]\\ &\leq (1+\frac{1}{\epsilon})W\log_2(1+W)\cdot\\ &\qquad m(1+W)\phi(2n^2+1)\cdot\ln(m(1+W)\phi\cdot n)\\ &\leq 8\cdot(1+\frac{1}{\epsilon})W^3\cdot\phi\cdot n^2\cdot m\cdot\ln(W\cdot\phi\cdot n\cdot m)\\ &= O(\frac{1}{\epsilon}W^3\cdot\phi\cdot n^2\cdot m\cdot\ln(W\cdot\phi\cdot n\cdot m)),\end{aligned}$$

which completes the proof of the theorem. ■

5 Conclusions

Our paper gives the first result on the weighted Shapley network design game with an affine linear cost function $c_e(w_e) = a_e w_e + b_e$. The first major result is the existence of $O(\log_2(W))$-APNE, which is shown by constructing an $O(\log_2(W))$-approximate potential function $\Phi(P) = \sum_{e\in E} c_e(w_e)\log_2(1+w_e)$. For the weighted Shapley network design game with constant cost function c_e, Chen and Roughgarden[16] proved the existence of $\log_2[e(1+w_{\max})]$-APNE. It can be seen from the comparison that the value of α becomes slightly larger, but is still within the acceptable range.

In addition, the computation of $(1+\epsilon)\alpha$-APNE in polynomial time by ϵ-best response dynamics is another major result. Since we can adopt the shortest-path search to ensure that each iteration step runs in polynomial time, we only need to prove that the expected number of iterations is polynomial in the parameters. Specifically, we assume that $\{a_e, b_e\}_{e\in E}$ are ϕ-smooth random variables and prove that the expected number of iterations is at most $O(\frac{1}{\epsilon}W^3\phi n^2 m\cdot\ln(W\phi nm))$, which completes the analysis on the time complexity of the ϵ-best response dynamics. For the unweighted Shapley network design game with ϕ-smooth random variables c_e, Giannakopoulos[18] proved that the expected number of iterations of the ϵ-best response dynamics is $O(\frac{1}{\epsilon}\phi\log(\phi)n\log^3(n)m^5)$. Then we can see that the power of m in the expected number of iterations is greatly reduced in our model.

In the future, there are plenty of issues with the model that are worth studying, such as the price of anarchy and the price of stability of the game. Furthermore, we can further generalize the cost function into a polynomial function or even a general continuous function and study the α-APNE of the game.

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