物理实验

A MORE ACCURATE INVESTIGATION INTO THE RELATIONSHIP OF ORIGINAL AIR VOLUME AND PRESSURE WITH CRITICAL DEPTH OF THE CARTESIAN DIVER

ZHU Siyin　ZHOU Shiyun
(Department of Physics, Fudan University, Shanghai 200433)

Abstract The purpose of this paper is to investigate the effect of the original air volume and air pressure on the critical depth of the Cartesian diver. Using a transparent cylindrical tube as a container and a plastic tube as the Cartesian diver, the linear relationship between the critical depth h_c and the volume of trapped air inside the diver V_{air0}, as well as the linear relationship between the critical depth h_c and the initial pressure inside long tube P_{tube}, were proven. In order to improve the accuracy of the experiment, the independent variable of V_{air0} must be carefully controlled. Therefore, we focused on the relationship between the critical depth h_c and the change of trapped air volume ΔV_{air0}, rather than the original V_{air0}, thereby improving the control accuracy from 1mL to 0.01mL by adding air to the Cartesian diver with a syringe.

Key words Cartesian diver; critical depth; improvement of experimental accuracy

更准确地探究浮沉子临界深度与其初始气体体积和初始气压关系的方法

诸思吟　周诗韵
(复旦大学物理学系,上海　200433)

摘　要 本研究的目的是研究初始气体体积和气压对浮沉子临界深度的影响。以透明长圆柱管为容器,塑料管为浮沉子,我们通过实验证明了临界深度 h_c 与浮沉子内部初始气体体积 V_{air0} 的线性关系,以及临界深度 h_c 与长管内初始气压 P_{tube} 的线性关系。为了提高实验的准确性,必须仔细控制自变量 V_{air0}。因此,我们重点关注临界深度 h_c 与初始气体体积变化量 ΔV_{air0} 之间的关系,而不是原来的 V_{air0},从而通过使用注射器向浮沉子内添加空气,将气体体积控制精度从 1mL 提高到了 0.01mL。

关键词 浮沉子;临界深度;实验精度提高

收稿日期: 2022-11-02

作者简介: ZHOU Shiyun, female, lecturer, mainly engaged in physics experiment teaching, research direction is physics experiment teaching, zhoushiyun@fudan.edu.cn。

引文格式: 诸思吟,周诗韵. 更准确地探究浮沉子临界深度与其初始气体体积和初始气压关系的方法[J]. 物理与工程,2023,33(3):57-63.

Cite this article: ZHU S Y, ZHOU S Y. A more accurate investigation into the relationship of original air volume and pressure with critical depth of the Cartesian diver[J]. Physics and Engineering, 2023, 33(3): 57-63.

The Cartesian diver, created as a classic science experiment which proves Archimedes' principle and the ideal gas law, is now widely used as a teaching aid[1]. A simple Cartesian diver consists of a little downward bottle with its lid open and some air trapped in. And the diver flows on the water surface in a big airtight container almost filled with water. When we increase the pressure inside the container, such as pressing the walls of the container, the little bottle sinks from the surface of the water. When the pressure changes back to its initial value, the little bottle may return or never return to water surface, determined by the kinematic velocity and depth of the Cartesian diver's movement when we change the pressure.

Plenty of apparatuses have been designed and fabricated to demonstrate this phenomenon. To better control and measure the initial pressure in the long tube, a demonstration apparatus was built with a transparent cylindrical tube and a flange cover connected with a digital manometer[2]. Some others focused on its teaching application and made it interesting, like using two kinds of liquids in the large bottle[3] or making it like a fishing game[4]. There are also articles focused on dynamic phenomenon, like the Cheerios effect[5] and the 'smoke ring' phenomenon with water flowing out[6].

In addition, the most attracting phenomenon is that sometimes the Cartesian diver can't return to the initial place even though the pressure inside the container changed back to its initial value. At present, theoretical researches have been done using the dynamic method[7,8]. Besides, the calculation of its potential energy has given us a better understanding of its returning condition[9]. It has been shown that in static equilibrium, the key criterion of diver's resurfacing or sinking is whether the diver's position reaches the critical depth, which depends on the initial pressure of the container, the amount of air trapped inside the diver and the diver's own properties such as its volume and mass, etc.

However, there are few experimental studies on the parameters affecting the critical depth. Therefore, a set of experimental instruments was developed, with variables specially designed to accurately probe the dependence of the critical depth on the volume of air trapped in diver and the initial pressure inside the tube.

1 Models of divers

A transparent cylindrical tube is used as the container and a downward test tube is used as the diver (as shown in Figure 1). There are both water and air inside the Cartesian diver at the water surface. When the pressure inside the tube increases, the air volume inside the diver decreases and the buoyancy declines, resulting in the sinking of the Cartesian diver. When the diver sinks to a certain depth, let's discuss whether it will return to the water surface after returning the pressure in the tube to its initial value.

To simplify the model, we only consider a diver with solid enclosure. This means that we do not consider the effect of its shell deformation on the air volume trapped inside. Since the moving of the diver is much slower than the change of the air volume trapped inside the diver during its movement, we only need to perform a static force analysis to determine its movement trend at a certain depth. Therefore, only gravity, buoy-

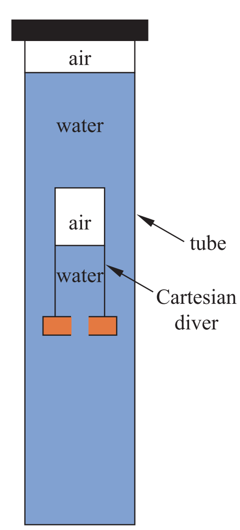

Figure 1 A draft of our model

ancy and surface tension should be considered and resistance exerted by water can be neglected (as shown in Figure 2). When the Cartesian diver balances at a certain depth, it will remain motionless. If the diver rests at a shallower place, the air volume in the diver will become larger as well as the buoyancy, so the diver will go up. With the diver going up, the air volume will increase further, so the diver will continue going up until it reaches the water surface. On the other hand, if the diver rests at a deeper place, the air volume in the diver will become smaller as well as the buoyancy, so the diver will go down.

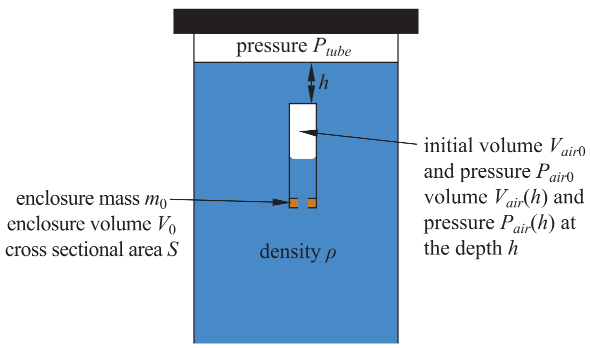

Figure 2 Variables used in calculation

Therefore, the key point is the depth at which the buoyancy is equal to the gravity plus surface tension, called the critical depth. If the diver's position exceeds the critical depth after we increase the pressure in the tube, it will not be able to return to the water surface even if the initial pressure is restored.

Some approximations are made to simplify the model and calculation for the critical depth. First, air is considered as ideal gas, as our experiments were done at room temperature and atmospheric pressure. Second, the mass of the trapped air, which is much smaller than the mass of the diver's enclosure, is neglected[9].

Using the ideal gas law for trapped air in the diver, we have

$$P_{air0}V_{air0}=P_{air}(h)V_{air}(h) \tag{1}$$

where P_{air0}, V_{air0} are the pressure and volume of trapped air under atmospheric pressure, respectively, and $P_{air}(h)$, $V_{air}(h)$ is the pressure and volume of trapped air in the diver at depth of h, respectively. h is defined as the distance from the top of the diver to the water surface when it is underwater, and P_{tube} is the pressure inside the tube, as shown in Figure 3.

Also, we have

$$P_{air}(h)=P_{tube}+\rho g\left(h+\frac{V_{air}(h)}{S}\right) \tag{2}$$

where P_{tube} is the pressure inside the tube, S is the internal cross section of the diver enclosure.

At the certain depth h, using Archimedes' principle for the diver, we have

$$F_f=\rho g(V_{air}(h)+V_0) \tag{3}$$

where F_f is buoyancy on the diver, $\rho=0.995\times 10^3\,\mathrm{kg/m^3}$ is the water density, $g=9.794\,\mathrm{m/s^2}$ is the gravitational acceleration at Shanghai, V_0 is the volume of the Cartesian diver's enclosure.

If the diver is just at the critical depth h_c, we have the force balance equation

$$F_f=m_0g+F_s \tag{4}$$

Where m_0 is the mass of the Cartesian diver's enclosure and F_s is the surface tension (as shown in Figure 3).

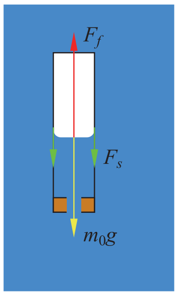

Figure 3 Force analysis of the Cartesian diver

With equations (1)～(4), we can get the expression of the critical depth h_c

$$h_c=\frac{P_{air0}V_{air0}}{m_0g+F_s-\rho gV_0}-\frac{P_{tube}}{\rho g}-\frac{m_0g+F_s-\rho gV_0}{\rho gS} \tag{5}$$

2 Experimental Setup

Noticing that the critical depth h_c has linear relationship with V_{air0} and P_{tube}, we can prove their relationship by experiment. The overall experimental setup is shown in Figure 4(a), and

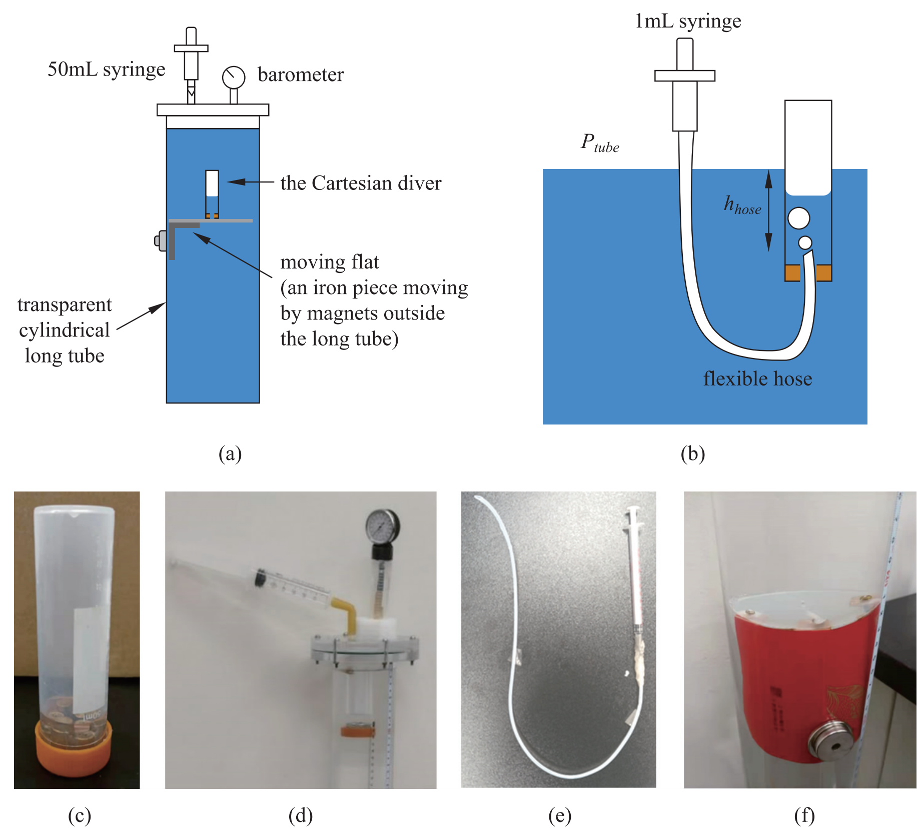

Figure 4 Experimental setup

(a) A sketch of experimental installation. (b) A sketch of 1mL syringe and a flexible hose. (c) The Cartesian diver used in experiments. (d) The long transparent tube, 50mL syringe and barometer. (e) The 1mL syringe and the flexible hose. (f) The moving platform made by an iron piece inside the long tube and magnets outside.

the used diver is shown in Figure 4(c).

It is a good way to use a large syringe to control the pressure inside the tube, and to show the value of pressure with a barometer, as shown in Figure 4(d). As the piston of syringe slides in or out, the air pressure inside the long tube increases or decreases. If the piston is held in a certain place, the air pressure will remain the same. In this way we can control P_{tube}.

If the diver can hover naturally at the critical depth, the value of h_c can be accurately measured. In experiment, however, this hovering rarely occurs because h_c is an unstable equilibrium point. Therefore, in our experiments, a horizontal platform that can move up and down in the tube was added to control the diver's position in the water, as shown in Figure 4(a) and 4(f). Apply a specific external pressure to the water surface to make the diver sink to the platform, then restore the pressure to its initial value, and observe the diver's movement. If the diver floats up at depth h_1 and sinks at depth h_2, the critical depth h_c at pressure P_{tube} is just between h_1 and h_2. That is, the critical depth $h_c=\frac{h_1+h_2}{2}\pm\frac{h_2-h_1}{2}$. By carefully adjusting the platform, the error bars of h_c could be narrowed to 2～4mm in our experiments.

Another problem is how to measure the initial volume of trapped air V_{air0} in the Cartesian diver. Unfortunately, V_{air0} significantly affects the critical depth h_c. Take the typical parameters in our experiments as an example: $P_{air0}=P_{tube}=1\text{atm}$, $m_0=50\text{g}$, $F_s=6\times10^{-3}\text{N}$, $V_0=20\text{mL}$, $S=5\times10^{-4}\ \text{m}^2$. If V_{air0} is 34mL, then h_c is 0.993m. But if V_{air0} *is* 33mL, then h_c is 0.656m. Therefore, the accuracy of volume V_{air0}

is very important in this experiment. In order to meet the requirement of a total air volume of about 40mL, we usually use a graduated cylinder with a total volume of 50mL to measure the injected air volume. However, its typical accuracy is only 1mL, which is much lower than the accuracy of 0.01ml required in this experiment.

Our solution is following. As what we need to verify is the linear relationship between h_c and V_{air0}, we can measure the volume change ΔV_{air0} rather than V_{air0} and prove the linear relationship between h_c and ΔV_{air0}. Let $V_{air0}=V'_{air0}+\Delta V_{air0}$. That is to say, what we need to verify is

$$h_c=\frac{P_{air0}}{m_0g+F_s-\rho gV_0}\Delta V_{air0}+\left(\frac{P_{air0}}{m_0g+F_s-\rho gV_0}V'_{air0}-\frac{P_{tube}}{\rho g}-\frac{m_0g+F_s-\rho gV_0}{\rho gS}\right) \tag{6}$$

To control and measure ΔV_{air0}, each time we use a syringe to obtain a certain amount of air at room temperature and atmospheric pressure, such as 0.30 mL, and inject it into the Cartesian diver through a flexible hose, as shown in Figure 4(b) and 4(e). In this way, a small syringe with a range of 1mL and the minimum scale value of 0.01mL can be used, which improves the reliability of the experimental verification of the linear relationship between h_c and V_{air0}. In our experiment, the error bars of ΔV_{air0} could be narrowed down to 0.01~0.03mL.

3 Results

With methods above, the linear relationship between the critical depth h_c and the volume of trapped air V_{air0} is proved (as shown in Figure 5). The fitting function is $h_c=32.1(\text{cm/mL})\Delta V_{air0}+16.8(\text{cm})$, and $R^2=0.9995$.

We can compare the experimental and theoretical results. A linear fit to the experimental data gives a slope of

$$k_1=(32.1\pm0.3)\text{cm/mL}$$

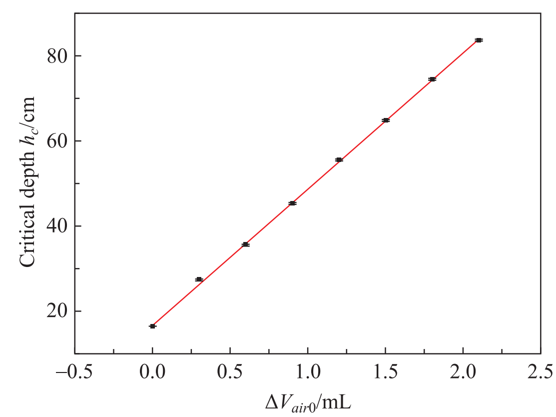

Figure 5 The linear dependence between the critical depth and the volume of air trapped inside the Cartesian diver

And from expression (5), the theoretical slope under the current experimental conditions is

$$k'_1=\frac{P_{air0}}{m_0g+F_s-\rho gV_0}=(33.3\pm0.5)\text{cm/mL}$$

Since there is only a 3.6% deviation between the experimental and theoretical slopes, the relationship between the critical depth and the volume of trapped air is proven.

Besides, the linear relationship between the critical depth h_c and the initial pressure inside long tube P_{tube} is also proved (as shown in Figure 6). The fitting function is $h_c=-10.7(\text{cm/kPa})P_{tube}+1.15\times10^3(\text{cm})$, and $R^2=0.998$.

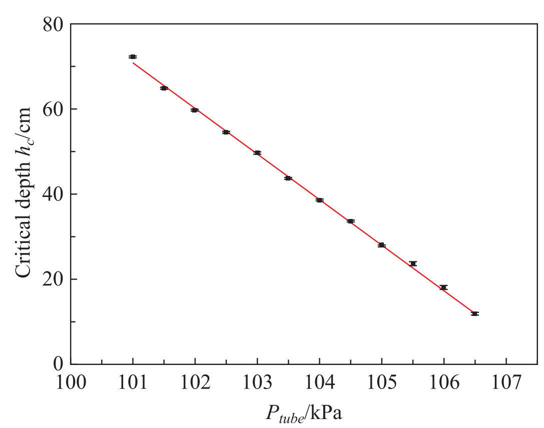

Figure 6 The linear dependence between the critical depth and the initial pressure inside long tube

A linear fit to the experimental data gives a slope of

$$k_2=(-10.3\pm0.2)\text{cm/kPa}$$

And from expression (5), we can compute the theoretical slope

$$k'_2 = -\frac{1}{\rho g} = (-10.7 \pm 0.3)\,\mathrm{cm/kPa}$$

Therefore, the linear relationship between the critical depth and the initial pressure is also demonstrated.

4 Discussion

We notice a 3.6% discrepancy between k_1 and k'_1. Here is one possible explanation. When air is injected, the port of flexible hose is below the water surface, causing the pressure at its port increases slightly compared with atmospheric pressure and resulting in a reduction of air volume that is actually injected into the diver.

In our experiment, the depth of the hose port is not more than 5cm, and the air imported by the syringe is expected to be 0.30ml for each injection, so the total air volume inside the syringe and the hose does not exceed 1mL. Let us take $h_{hose} \approx 0.05$m and $V_{s+h} \approx 1$mL to estimate the deviation of each imported air. The pressure inside the syringe is $P_{tube} + \rho g h_{hose} \approx 1.015 \times 10^5$Pa. Therefore, the deviation of the imported air volume (that is, the deviation of ΔV_{air0} each time) is $\delta V = \frac{P_{tube} V_{s+h}}{P_{tube} + \rho g h} - V_{s+h} \approx 0.005$mL.

According to the above estimation, if the actual imported air per injection is smaller by 0.005mL, the obtained slope of the fitting curve in figure 5 will be at most 2cm/mL smaller, which is 6% smaller than the theoretical slope k'_1. This is consistent with our experimental results.

Besides, we can imagine that if the internal diameter of the hose becomes smaller, the deviation of the slope will reduce as well due to the decrease of V_{s+h}. In our experiment, we used the thinnest possible hose with the internal diameter of 1mm, since the needle on the syringe is only slightly less than 1mm, and the length of the hose is as short as possible. To prove our speculation, we repeated our work using a flexible hose with 2mm of inner diameter and the same length. As we expected, the slope of the fitting curve to the new data is (29.9±0.2)cm/mL, which deviates more from the value of theoretical slope k'_1.

5 Conclusions

In the research, we experimentally proved the relationship between the critical depth h_c of the Cartesian diver and its initial conditions, which was only studied theoretically in the past. Under the presumption of static equilibrium, experimental evidences show that the critical depth h_c of the Cartesian diver is linearly related to the initial volume of trapped air V_{air0}, and also to the initial pressure above the water surface P_{tube}. It should be notified that our experiment was done at room temperature and atmospheric pressure, so it is reasonable to use ideal gas law. If not, the relationship between h_c and V_{air0} or P_{tube} will show deviation from linearity.

Considering that the accuracy of measuring V_{air0} is the main factor affecting the reliability of the proof of linear relationship here, we paid our attention to the air volume change ΔV_{air0} rather than the total air volume V_{air0}, and used a small syringe with a minimum division of 0.01ml to add air into the Cartesian diver, which improved the accuracy of our result. As shown in the estimation in Experimental Setup, the error of the independent variable itself was decreased from about 1mL to about 0.01mL. In this way, the relationship between the critical depth h_c and the initial volume of trapped air V_{air0} can be studied more accurately.

In scientific research, trying to improve the control accuracy of independent variables can obtain more convincing experimental results. The method of verifying the linear relationship by replacing the independent variable itself with the amount of change of the independent variable can be used in other experiments. Also, the method of using upper and lower limits to determine the unstable equilibrium point can also provide implications for other experimental designs.

References

[1] WIKIPEDIA. Cartesian diver [EB/OJ]. (2021-9-13)[2021-9-20]. https://en.wikipedia.org/wiki/Cartesian_diver
[2] GÜÉMEZ J, FIOLHAIS C, FIOLHAIS M. A Demonstration Apparatus for the Cartesian Diver[J]. *The Physics Teacher*, 2003, 41(8): 495-496.
[3] PLANINSIC G, KOS M, JERMAN R. Two-liquid Cartesian diver[J]. *Physics Education*, 2004, 39(1): 58-64.
[4] AMIR N, SUBRAMANIAM R. Making a fun Cartesian diver: a simple project to engage kinaesthetic learners[J]. *Physics Education*, 2007, 42(5): 478.
[5] CHEN C, LEE W, KAO S. The Cartesian diver, surface tension and the Cheerios effect[J]. *Physics Education*, 2014, 49(1): 50-54.
[6] ACKERSON B. Cartesian Diver Plus[J]. *The Physics Teacher*, 2020, 58(2): 84-85.
[7] IVANOV D, NIKOLOV S. Once more about the Cartesian diver[J]. *Physics Education*, 2020, 55(2): 1.
[8] DE LUCA R, GANCI S. A lot of good physics in the Cartesian diver[J]. *Physics Education*, 2011, 46(5): 528.
[9] GÜÉMEZ J, FIOLHAIS C, FIOLHAIS M. The Cartesian diver and the fold catastrophe[J]. *American Journal of Physics*, 2002, 70(7): 710-714.

■

(上接第 48 页)

时候,如何升维以及升维多少又是另外两个非常关键的问题,其解决方案依赖于对物理问题的深刻认识,有时也依靠直觉和运气。

本文通过讨论物理学中对维度改变求解问题的现象,明确地将“降维打击”这一概念引入物理学教学过程中,给学生种下发散思维的种子,以期提高学生的创新、思辨等能力,同时在物理学教育同行中起到抛砖引玉的作用,助力课堂教学。

参 考 文 献

[1] 王蒙,陶俊琦,程剑剑,等. 物理学中常用的高斯与类高斯型积分[J]. 大学物理,2021,40(5): 13-19.
WANG M, TAO J Q, CHENG J J, et al. Gaussian and Gaussian-like integrals in physics[J]. College Physics, 2021, 40(5): 13-19. (in Chinese)
[2] 曾谨言. 量子力学(卷一,卷二)[M]. 北京:科学出版社, 2013.
[3] 周世勋. 量子力学教程[M]. 北京:高等教育出版社,2009.
[4] 同济大学数学系. 线性代数[M]. 北京:高等教育出版社, 2007.
[5] 倪致祥. 厄米本征值问题的探究[J]. 大学物理,2008, 27(2): 39-41.
NI Z X. Explorating for the Hermite eigenvalue problem [J]. College Physics, 2008, 27(2): 39-41. (in Chinese)
[6] 胡先权,王帮美. 也谈厄米多项式的递推关系[J]. 大学物理,2009,28(6): 22-23.
HU X Q, WANG B M. Discussing on the recurrence relation of the Hermite polynomial also[J]. College Physics, 2009, 28(6): 22-23. (in Chinese)
[7] 吴崇试. 厄米多项式的定义及其他[J]. 大学物理,2010,29(3): 14-15.
WU C S. Two kinds of the Hermite polynomials[J]. College Physics, 2010, 29(3): 14-15. (in Chinese)
[8] ARFKEN G B, WEBER H J, HARRIS F E. 物理学家用的数学方法[M]. 北京:世界图书出版公司,2014.
[9] 王竹溪. 特殊函数概论[M]. 北京:科学出版社, 1963.

■