

Analytical solution of ball-disc oil film capacitance for various lubrication conditions

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Abstract

Accurately solving the electrical impedance in various conditions is fundamental in monitoring the lubrication conditions of rolling bearings with electrical impedance method. In the present study, an analytical solution using *Legendre* polynomials for the capacitance of ball-disc contact is proposed, accounting for the clearance, oil reservoir angle, oil-solid contact angle, Hertz deformation, air capacitance, and extending to dynamic oil reservoir. The analytical results show good agreement with the numerical simulation results from finite element model and ball-disc experiment with specific oil reservoirs and dynamic conditions. Parametric studies reveal that the oil reservoir angle, indicating the oil supply condition, should be considered, especially for larger film thickness and fully-flooded lubrication. The capacitance contributed by air should be involved because the dielectric constants of oil and air are generally close, while the

effect of contact angle is negligible for common practical conditions.

Keywords: Ball-disc model; Oil film capacitance; EHD lubrication; Analytical solution

1. Introduction

Rolling bearings are essential components in mechanical engineering and are widely used in various mechanical systems. Typically, rolling bearings operate under the elastohydrodynamic lubrication (EHD) conditions [1-3]. Lubricant starvation of the contact area may occur in high-speed or grease-lubricated bearings [4], reducing equipment reliability. Therefore, dynamically monitoring lubrication condition of bearings is crucial. Oil film thickness, as the most direct and critical indicator, is commonly used to assess the lubrication state of bearings.

Several methods have been proposed for non-destructive monitoring of oil film thickness in rolling bearings [5], such as optical, ultrasonic, and electrical measurements. Several classical optical methods have been proposed to measure oil film thickness with high resolution, such as the spacer layer imaging method [6], the CIELAB method proposed by Hartl [7], optical relative interference method [8], and the multi-beam intensity-based method [9]. Many studies, such as Liang et al. [10], have conducted experimental measurements using optical interference method and observed oil film thickness and shape under different rotational speeds. However, optical methods have some limitations: bearing materials are typically opaque, the frictional scratches or surface roughness can affect the interference images, and the complex algorithms and instruments are often required [11, 12]. The ultrasonic method, a non-contact method for measuring oil film thickness [13, 14], is applicable to various materials, including opaque materials. However, the thermal sensitivity of ultrasonic signals and difficulties in aligning the focal area with the position of minimum oil film thickness can lead to significant measurement errors [15].

Compared to the above methods, electrical methods offer greater convenience and enable real-time measurement of oil film thickness under actual working conditions, as

they require simpler measuring equipment and provide faster responses in electrical signals. Electrical methods are mainly categorized into capacitance and impedance methods. The capacitance method estimates the oil film thickness by evaluating the capacitance of the lubricated area. Crook [16, 17] assumed that the inlet of Hertz contact region between two balls is filled with oil, while the outlet contains two equally thick oil layers close to the surfaces of the balls, with air between them. Dyson et al. [18] suggested that the Hertz contact region can be considered as a parallel plate capacitor due to the smooth, flat central region. Based on this equivalence, Chua et al. [19] solved the capacitance inside and outside the Hertz contact area by equating the total capacitance to the parallel connection of multiple parallel plate capacitors. Jablonka et al. [20, 21] discovered that the oil around the Hertz contact region can be divided into three regions: the outside, EHD contact, and cavitation regions, based on which they considered the effect of air when calculating the capacitance of the cavitation region and also performed experimental verification. Shetty et al. [22] extended Jablonka's capacitance calculation formula by considering the morphology of oil film under oil starvation conditions. Hamrock and Dowson [23] established a relationship between the minimum oil film thickness and various parameters, such as velocity, load, and material properties, which facilitated the calculation of capacitance in the Hertz contact region. For example, many researchers have considered that the total capacitance in the oil-immersed region is equal to the product of a coefficient and the capacitance in the Hertz contact region [24-27].

Under EHD lubrication, contacts may also experience oil starvation or oil film breakdown [4], both of which can be assessed through the contact resistance measurements [28]. The electrical impedance method can simultaneously measure both capacitance and resistance. Electrical impedance-based monitoring enhances data quality and has attracted increasing research interest [29]. By applying an AC voltage and measuring the current's amplitude and phase, this method can evaluate multiple variables simultaneously. Nakano et al. [30] established a "steel-oil-mercury" system and investigated the variation of both oil film thickness and the ratio of the contact area covered by the oil film using the resistance method. Schnabel et al. [31] applied the

electrical impedance method to derive capacitance and examined how surface roughness affects contact capacitance and resistance. Maruyama et al. [32-34] proposed a formula for calculating the capacitance in the ball-on-disc EHD contacts and further extended it to bearing applications. Assuming the lower hemisphere of the ball is completely immersed in oil, they calculated the capacitance by the equivalent method of integrating the capacitance of the parallel plates over the ball-disc gap. With the measurement of impedance amplitude and phase, they inversely derived the breakdown ratio and the oil film thickness, achieving good agreement with their experimental results. Based on this method, Manjunath et al. [35] used the electrical impedance spectroscopy (EIS) method to investigate the electrical impedance characteristics of the EHD contact area across different frequencies. Yu et al. [36] validated that EIS is suitable for identifying the circuit models that indicate different lubrication conditions.

The previous capacitance models typically follow two main approaches: integrating the parallel-plate capacitor model or applying correction factors to the Hertzian capacitance, both of which rely on simplified geometric assumptions and often require approximations or empirical adjustments. Generally, the dielectric constant of air is not small compared to that of oil. Therefore, the air between ball and disc significantly contributes to the total capacitance and cannot be neglected. In addition, the meniscus formed by oil between ball and disc surface, due to contact angle of oil, also affects the capacitance. In this study, an analytical capacitance expression is derived based on Laplace's equation for the electric potential and charge distribution between the ball and disc. The analytical solution accounts for the contributions of the oil reservoir and air, as well as minimum clearance and Hertzian deformation, while the contact angle effect is incorporated through empirical fitting to further improve the model accuracy and applicability. Besides, the model can be extended to ball bearings and provides a valuable reference for bearing oil film capacitance calculations and impedance-based lubrication monitoring.

2. Methodology

To monitor the lubrication performance of ball bearings, it is essential to establish

a model that accounts for various oil film conditions. This section presents the proposed model and the solution method for the capacitance of EHL contact area.

2.1 Basic model and bi-spherical coordinate system

Fig. 1(a) illustrates the contact between ball and raceways for a ball bearing with lubrication. To simplify the calculations, the ball-raceway contact is modeled as a ball-disc system and the oil distribution is assumed to be axisymmetric. The contact region shown in Fig. 1(b) can be divided into three concentric regions: an outer air region ($\theta > \theta_1$); an oil reservoir region ($\theta_2 > \theta > \theta_1$, including the meniscus region and surround region); and a Hertz deformed region ($\theta < \theta_2$, including the lubricated region, with film thickness h , and a breakdown region). If there are several contact spots in the Hertz deformed region shown in Fig. 1(c) due to the surface roughness, they can be equivalently represented as a circular area $m \cdot \pi a^2$ at the center of the Hertz deformed region, where m is the breakdown ratio and a is the Hertz contact radius. In the EHD lubrication model between a single ball and the bearing raceway, the capacitance contributed by the air region and the oil reservoir region is denoted as C_1 and C_2 , respectively. The circuit formed by Hertz deformed region can be divided into two parts: C_3 from the lubricated region and R_1 from the breakdown region. All the capacitors and resistors of each part given above are connected in parallel, and the equivalent circuit diagram of ball-disc EHD contact is shown in Fig. 1(d).

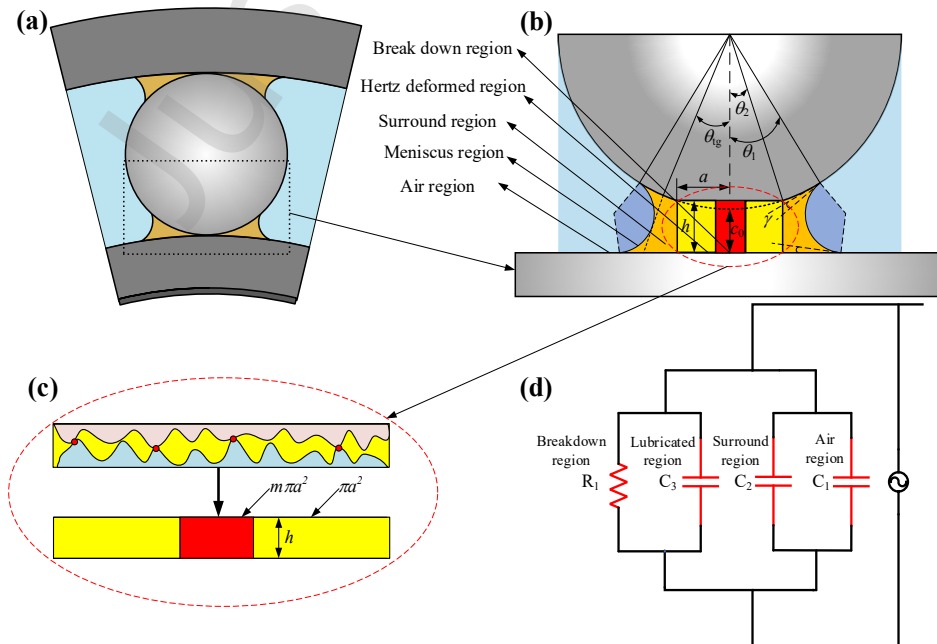


Fig. 1. Geometric model and circuit diagram of ball-disc model (a) contact between ball and raceways with lubrication (b) schematic diagram of the EHD area (c) schematic diagram of Hertz deformed region (d) circuit diagram of ball-disc EHD contact

According to Fig. 1(d), the total impedance can be obtained as follows:

$$Z_1 = \frac{R_1}{1 + j\omega R_1 (C_1 + C_2 + C_3)} \quad (1)$$

where ω represents the angular frequency of alternating current.

To accurately calculate the capacitance shown in Fig. 1(b), the problem must be approached by solving the potential distribution between the ball and disc. For this purpose, the ball-disc system is modeled as an eccentric spherical shell. This reformulates the problem as an electrical problem in the bi-spherical coordinate system, a three-dimensional orthogonal coordinate system defined by ξ , η , and ϕ , whose projection on the YOZ plane is shown as the dotted line in Fig. 2(a). It is commonly used to solve electric field problems involving two spherical conductors of different radii [37, 38]. In this system, as shown in Fig. 2(a), $\xi = \xi_1$ corresponds to the spherical surface of radius r_1 , and $\xi = \xi_2$ to radius r_2 . The coordinate η represents the angle $\angle F_1 P F_2$, formed between a point P on the sphere and the two foci F_1 and F_2 . The coordinate ϕ represents the angle of the circle's rotation about the Z-axis. Thus, a sphere with radius r_1 can be represented by $\xi = \xi_1$ in bi-spherical coordinates as (ξ, η, ϕ) . As shown in Fig. 2(b), when the radius r_2 of the outer spherical shell gradually increases and approaches infinity, the model approaches a ball-disc model, as shown in Fig. 2(c). Based on the geometric relationship in Fig. 2(a), the half focal length, $l = |F_1 F_2|/2$, of the bi-spherical coordinate system is expressed as $l = r_1 \cdot \sinh \xi_1$, and the coordinates of the centers of the two spherical shells are $z_1 = l \cdot \coth \xi_1$, $z_2 = l \cdot \coth \xi_2$, where $\sinh$ and $\coth$ denote the hyperbolic sine and cotangent function, respectively. The relationship among the minimum clearance, eccentricity, and radius is given by:

$$r_2 = r_1 + e + c \quad (2)$$

where e is the eccentricity between two spherical shells. According to the cosine theorem, the relationship between ξ and r can be expressed as follows:

$$\cosh \xi_1 = \frac{r_2^2 - e^2 - r_1^2}{2r_1 e}, \quad \cosh \xi_2 = \frac{r_2^2 + e^2 - r_1^2}{2r_1 e} \quad (3)$$

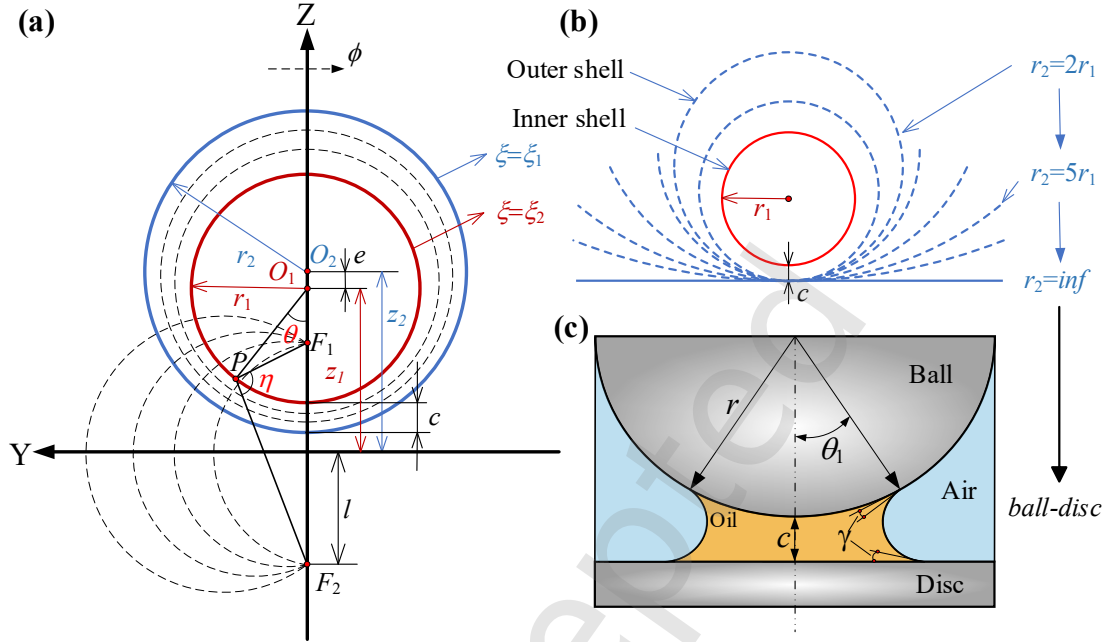


Fig. 2. Schematic diagram of the eccentric spherical shell and its evolution: (a) coordinate parameters of the bi-spherical coordinate system (in the YOZ plane) (b) degradation from eccentric spherical shell to ball disc (c) ball-disc model

Based on Eq. (2) and geometric relationship in Fig. 2(a), the relationship between θ in the rectangular coordinate system and η in the bi-spherical coordinate system is derived as follows:

$$2e^\xi \cos \eta = e^{2\xi} + 1 - \frac{4l^2}{r_1^2 + (z-l)^2 - 2(z-l)r_1 \cos \theta} \quad (4)$$

2.2 Solution of electrical potential and capacitance

Capacitance is defined as the amount of electric charge stored on conductors per unit potential difference between them. Evaluating it involves first solving the potential distribution, followed by calculating the surface charge density and integrating it to obtain the total charge, and finally solving the capacitance. Solving Laplace's equation is the common method to obtain the potential. The Laplace's equation in the bi-spherical coordinate system is shown in Eq. (5), which can be transformed from Laplace's equation in the Cartesian coordinate system:

$$\frac{1}{h_\xi^2} \frac{\partial}{\partial \xi} (h_\xi^2 \frac{\partial \phi}{\partial \xi}) + \frac{1}{h_\eta^2} \frac{\partial}{\partial \eta} (h_\eta^2 \frac{\partial \phi}{\partial \eta}) + \frac{1}{h_\phi^2} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \quad (5)$$

By substituting *Lamé* coefficients into Laplace's equation in bi-spherical coordinate system:

$$h_\xi = h_\eta = \frac{l}{\cosh \xi - \cos \eta}, \quad h_\phi = \frac{l \sin \eta}{\cosh \xi - \cos \eta} \quad (6)$$

Laplace's equation in the bi-spherical coordinate system can be expressed as:

$$\frac{\partial}{\partial \xi} \left(\frac{\sin \eta}{\cosh \xi - \cos \eta} \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{\sin \eta}{\cosh \xi - \cos \eta} \frac{\partial \phi}{\partial \eta} \right) + \frac{1}{\sin \eta (\cosh \xi - \cos \eta)} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \quad (7)$$

By applying the transformation to Eq. (7) and considering the axial symmetry, Laplace's equation in the bi-spherical coordinate system can be simplified as follows, which is about variable ψ :

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \cot \eta \frac{\partial \psi}{\partial \eta} - \frac{1}{4} \psi = 0 \quad (8)$$

By separating the variables and solving this equation, the solution of Laplace's equation in the region of two spherical shells can be given by:

$$\phi(\xi, \eta) = V \sqrt{2(\cosh \xi - \cos \eta)} \sum_{n=0}^{\infty} \left[A_n \cosh\left(n + \frac{1}{2}\right) \xi + B_n \sinh\left(n + \frac{1}{2}\right) \xi \right] P_n(\cos \eta) \quad (9)$$

where A_n and B_n are the coefficients related to boundary conditions, $P_n(\cos \eta)$ is *Legendre* polynomial related to η .

The boundary conditions can be obtained by applying a potential of V to the inner sphere and by grounding the outer sphere to the earth:

$$\phi(\xi_1, \eta) = V, \quad \phi(\xi_2, \eta) = 0 \quad (10)$$

By substituting Eq. (10) into Eq. (9) and simplifying the equation, the coefficients A_n , B_n , and the solution of Laplace's equation can be obtained as follows:

$$A_n = V \frac{\sinh\left(n + \frac{1}{2}\right) \xi_2}{\sinh\left(n + \frac{1}{2}\right) (\xi_2 - \xi_1)} e^{-(n + \frac{1}{2}) \xi_1} \quad (11)$$

$$B_n = V \frac{\cosh(n + \frac{1}{2})\xi_2}{\sinh(n + \frac{1}{2})(\xi_1 - \xi_2)} e^{-(n + \frac{1}{2})\xi_1} \quad (12)$$

$$\varphi(\xi, \eta) = V \sqrt{2(\cosh \xi - \cos \eta)} \sum_{n=0}^{\infty} \frac{\sinh(n + \frac{1}{2})(\xi - \xi_2)}{\sinh(n + \frac{1}{2})(\xi_1 - \xi_2)} e^{-(n + \frac{1}{2})\xi_1} P_n(\cos \eta) \quad (13)$$

then, the surface charge on the ξ_1 -sphere is obtained as follows:

$$Q = \varepsilon \int_S (-\nabla \varphi) dS = \varepsilon \int_0^{2\pi} \int_{\eta_1}^{\eta_2} \frac{1}{h_\xi} \frac{\partial \varphi}{\partial \xi} h_\eta h_\phi \bigg|_{\xi=\xi_1} d\eta d\phi \quad (14)$$

Combining Eqs. (13), (14) and $C = Q/U$, where U represents the potential difference calculated from the expression of capacitance in bi-spherical coordinate system can be obtained. To address the presence of different dielectric constants in different regions (oil and air), the spatial distribution of each medium is represented using different integration bounds. Based on the relationship between η and θ derived in Eq. (4), the parameters η_1 and η_2 can be transformed into integration upper limits θ_{UL} and lower limits θ_{LL} , i.e., the angle in the Cartesian coordinate system. According to the previous assumption, r_2 should be close to infinity to represent a disc, and in the expression, this is considered by taking r_2 much larger than r_1 , thus the capacitance of ball-disc model as a function of c , r_1 , θ and ε can be obtained by combining Eqs. (2), (3), (4), (13) and (14), and simply represented as f :

$$f(c, \theta_{LL}, \theta_{UL}, r_1, \varepsilon) = \sqrt{2\pi\varepsilon l} \int_{\cos \theta_{UL}}^{\cos \theta_{LL}} \sum_{n=0}^{\infty} e^{-(n + \frac{1}{2})\xi_1} \left[\frac{\sinh \xi_1}{(\cosh \xi - \cos \eta)^{3/2}} + \frac{(2n+1)\coth(n + \frac{1}{2})(\xi_1 - \xi_2)}{(\cosh \xi - \cos \eta)^{1/2}} \right] P_n(\cos \eta) d(\cos \eta) \quad (15)$$

The dependency of C on pressure can be included by using C as $C(p)$, when its change causes a nonnegligible variation of C .

The integral in Eq. (15) is evaluated numerically using a truncated series expansion of *Legendre* polynomials. The convergence of the series with the number of terms and the calculation time T_c under three typical cases is shown in Fig. 3. As seen

from Fig. 3, the series converges rapidly, with negligible changes in the computed capacitance (relative variation below 0.01%) after $N = 50$. The average T_c is 22 ms for $N = 50$, which is used for all the calculations.

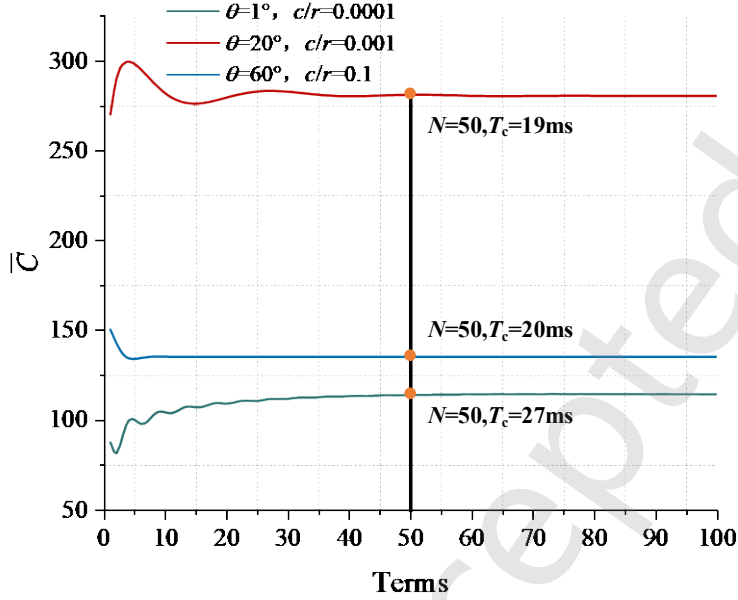


Fig. 3 The convergence of the series with the number of terms and the calculation time for typical cases

2.3 Consideration of air contribution and EHD lubrication conditions

Both the lubricant and the air will contribute to capacitance. Meanwhile, the EHD lubrication prevails in real bearing operation [39], where the ball will experience deformation and the thin lubricant film may rupture.

The capacitance contributed by the air region and the oil reservoir region (see Fig. 1(b)) can be calculated by adopting the integration limits in Eq. (15), from θ_1 to π for C_1 and from θ_2 to θ_1 for C_2 . It should be noted that when calculating C_1 and C_2 , which are outside the Hertz deformed area, the clearance c in Eq. (15) between the original ball and disc should be c_0 . According to the geometric relationship shown in Fig. 1(b), θ_2 and c_0 can be expressed as follows:

$$\theta_2 = \arcsin\left(\frac{a}{r_1}\right) \quad (16)$$

$$c_0 = h - r_1(1 - \cos \theta_2) \quad (17)$$

It should be noted that the influence of contact angle θ in the meniscus region on capacitance is not considered in this calculation, and its effect will be discussed in Sec. 2.5. Therefore, by considering meniscus region as a part of oil reservoir region, C_1 and C_2 can be obtained by Eqs. (18) and (19).

$$C_1 = f(c_0, \theta_1, \pi, r_1, \varepsilon_{\text{air}}) \quad (18)$$

$$C_2 = f(c_0, \theta_2, \theta_1, r_1, \varepsilon_{\text{oil}}) \quad (19)$$

Regarding the resistance in the breakdown region, based on Holm's equation, if R_{10} denotes the resistance for total breakdown (i.e., dry contact), the resistance for the entire breakdown region is $R_1 = R_{10}/\sqrt{m}$. The lubricated portion of the Hertz deformed region, with an area of $(1-m)\pi a^2$ and film thickness h , can be modeled as a parallel plate capacitor C_3 (the surface roughness is neglected). Therefore, C_3 and R_1 are given by:

$$C_3 = \varepsilon_{\text{oil}} \frac{(1-m)\pi a^2}{h} \quad (20)$$

$$R_1 = \frac{\rho}{4a\sqrt{m}} \quad (21)$$

where ρ represents the electrical resistivity of the metal of contact pairs.

2.4 Validation with the finite element model

The proposed method above is able to consider parameters including oil reservoir angle $\theta_{\square}$, minimum clearance c (or film thickness h), the radius of Hertz deformed region a , and the influence of air. The contact angle is not considered due to its complexity in analytical derivation, and the results in this case are compared with the finite element (FE) simulations, as shown in Fig. 7.

A FE model of the ball-disc lubrication system is developed in COMSOL Multiphysics® using the 2D axisymmetric Electrostatics module, as shown in Fig. 4 with its mesh design. The simulation conditions include $\theta_{\square}$ ranging from 5° to 60° , $c/r_1 = 0.001$ ($r_1 = 1\text{mm}$), $\varepsilon_{\text{oil}} = 10\varepsilon_{\text{air}}$ (with ε_{air} set as unit), $a = 0$ and $\theta_{\square} \square \square \square \square^\circ$. Since the applied potential difference across the ball-disc system does not affect its capacitance, a unit voltage is applied to the ball surface while the disc is grounded. The capacitance

$\bar{C}$ is normalized as follows:

$$\bar{C} = \frac{1}{\epsilon_{air} \cdot r_1} \cdot C \quad (22)$$

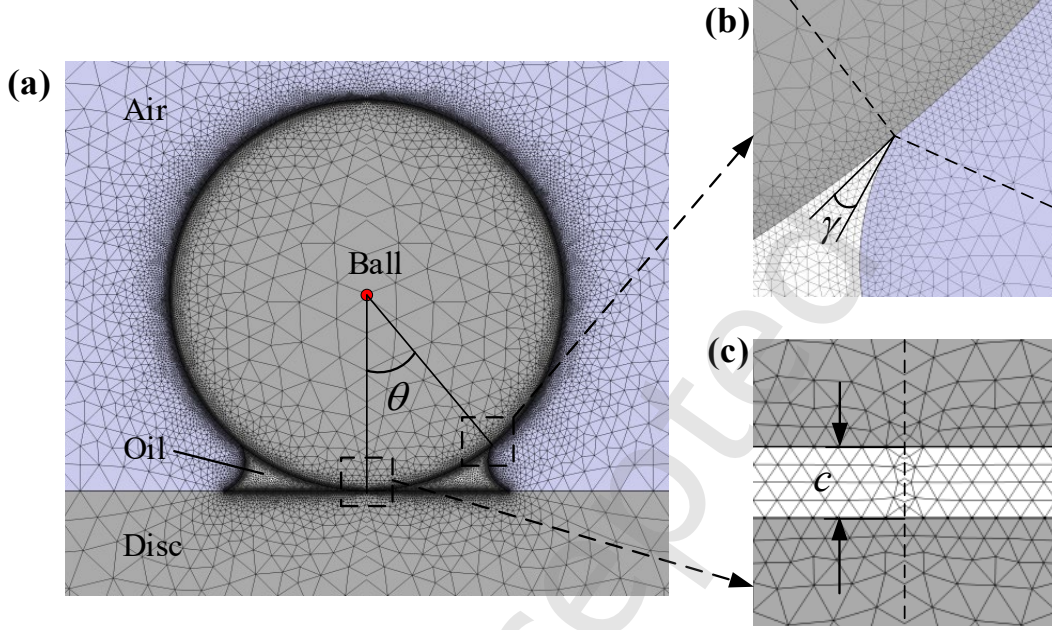


Fig. 4. The present model and its mesh design:(a) global view, (b) the mesh near contact angle γ and (c) the mesh near the minimum clearance c

A mesh convergence study was conducted to validate the FE model accuracy. The capacitance and relative errors for different mesh densities are presented in Fig. 5, showing rapid convergence, with the error dropping below 0.1% at approximately 3000 free tetrahedral elements. To further improve accuracy, local mesh refinement was applied to the minimum ball–disc gap, surfaces, and the oil–air interface. In addition, the adaptive refinement was employed to ensure solution accuracy for different cases. The final simulation results shown in Fig. 5 were obtained using approximately 180,000 elements, as shown in Fig. 4(c). 10 mesh elements were assigned across the minimum ball–disc gap in all cases, while the remaining mesh was adaptively generated according to geometry.

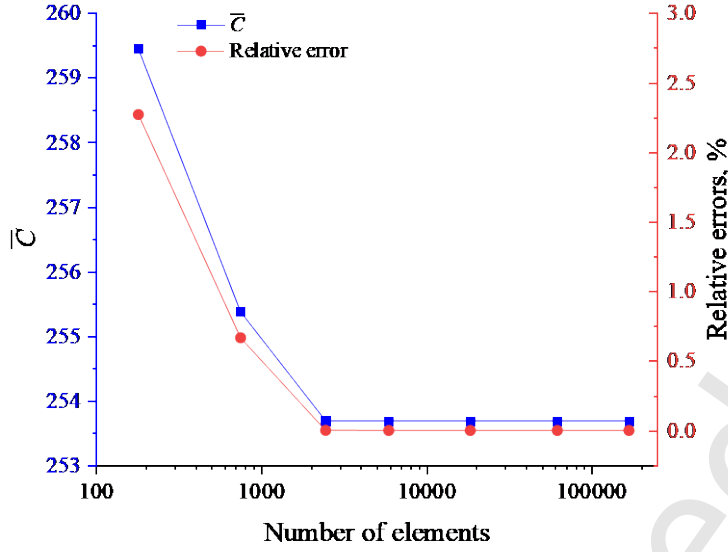


Fig. 5 Capacitance and error of different number of elements

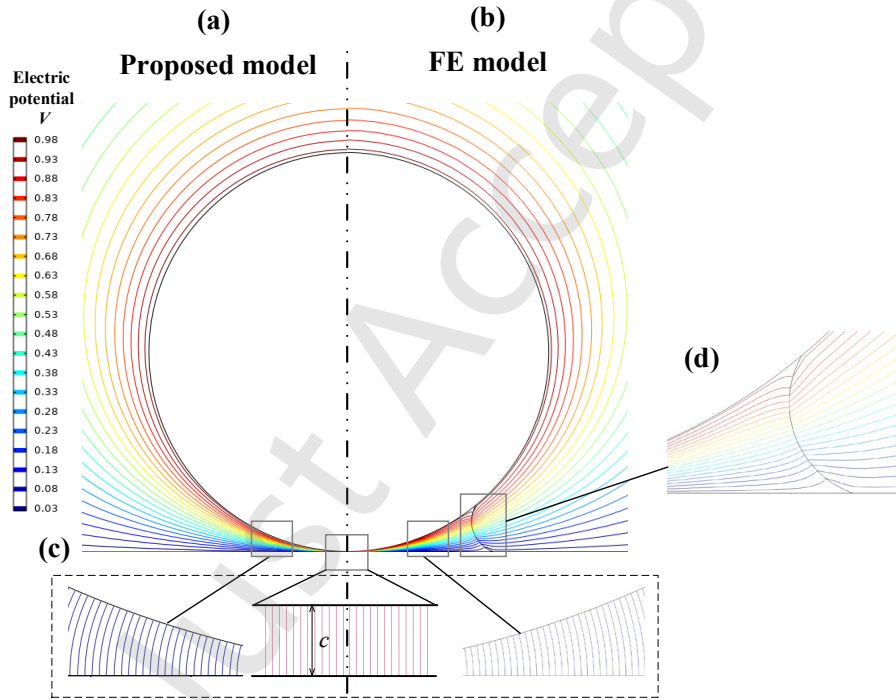


Fig. 6 Comparison of electric potential and electric field distributions between the proposed model and the FE model (a) electric potential obtained by proposed model; (b) electric potential obtained by FE model; (c) electric field obtained by proposed model(left) and FE model(right); (d) electric potential near the interface of oil and air

The potential and electric field distributions calculated by the proposed model and FE model are shown in Fig. 6. As shown in Fig. 6(a) and (b), the potential distributions obtained from these two models exhibit a consistent eccentric spherical pattern. The

electric field distributions along the disc centerline are shown in Fig. 6(c), with the proposed model on the left and FE model on the right. Near the minimum film thickness, the electric field is normal to the disc surface, while increasing curvature is observed toward the oil reservoir boundary. The approximation of parallel plate capacitor is suitable for the former, while for latter, this approximation inevitably introduces errors, especially at a larger oil reservoir angle θ_0 . It should be noted that the proposed model does not apply to the non-smooth equipotential lines near the oil-air interface, as shown in Fig. 6(d); this issue is considered in Sec. 2.5.

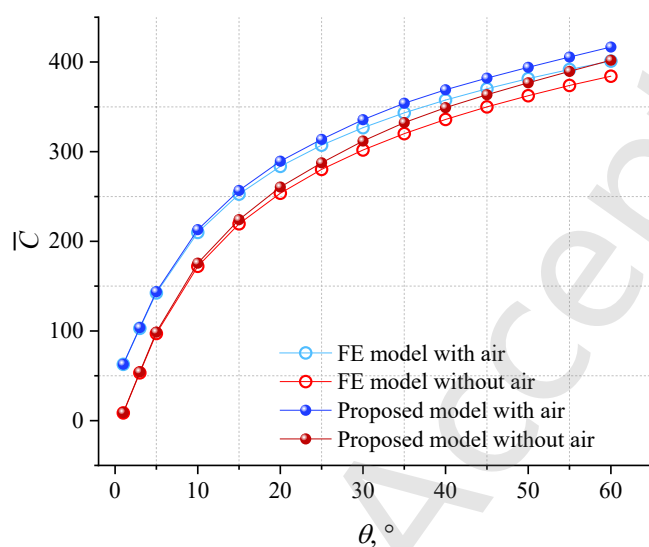


Fig. 7. $\bar{C}$ under various conditions

To evaluate the accuracy of the present method, the relative error e_θ is defined between the calculated and simulated values at various oil reservoir angles. To show the contribution of air, the dielectric constant of air is set to a very small value for the “without air” simulations. It is evident from Fig. 7 that the results from the proposed method exhibit good agreement with the FE simulation results, regardless of whether the effect of air is considered, showing a maximum e_θ of approximately 4% under the given conditions.

2.5 Consideration of contact angle effects

The error of 4% in the proposed method in Fig. 7 may be attributed to the lubricant distribution in the meniscus region (by the contact angle). As shown in Fig. 8, this region is simplified as a circular arc centered at (x_1, y_1) with a radius of r , which is

tangent to the ball and disc at a contact angle γ . θ_{tg} is the angle between the tangent line of the arc through the ball center and the Z-axis. In this region, as shown in Fig. 8(b), the oil is not strictly aligned with the potential gradient between the ball and the disc. When integrating to calculate the charge, passing through a region with a sudden change in charge can lead to errors.

To address this problem, the boundary between oil and air must be demarcated. If the integration range of oil is limited to (θ_2, θ_{tg}) , the oil contribution is less estimated, yielding a smaller capacitance. If the oil integration range extends to θ_1 , the oil medium is overestimated, yielding a larger capacitance. The dielectric constant in the meniscus region can be equivalently calculated by the weighted average of the dielectric constants of oil and air and their corresponding volumes. In the present situation, it is difficult to accurately calculate the volume ratio of oil and air. To simplify the volume calculation, a new computational strategy is introduced, using a corrected capacitance C' as the linear combination of the two calculation approaches, each based on a different choice of integration boundaries (θ_{tg} and θ_1). The oil capacitance and its corresponding integration limits are defined as C_{tg2} for (θ_2, θ_{tg}) and C_2 for (θ_2, θ_1) , while the corresponding air capacitance and its integration limits are: C_{tg1} for (θ_{tg}, π) and C_1 for (θ_1, π) , respectively.

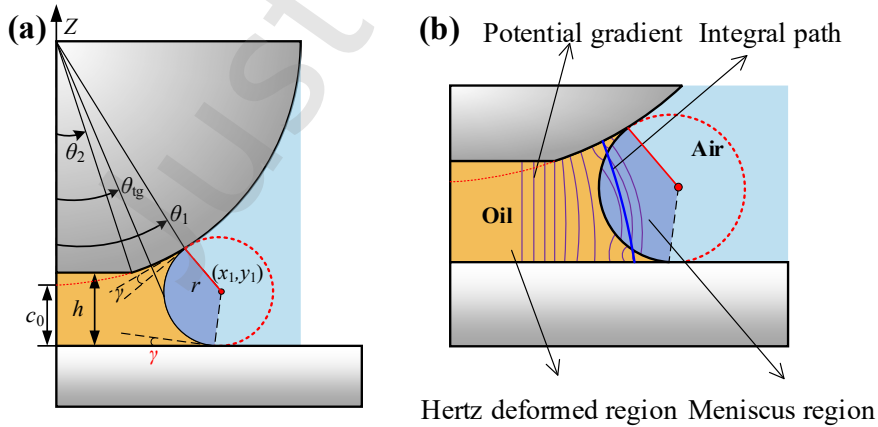


Fig. 8. Equivalent schematic diagram of oil and air distribution near the meniscus region (a) the arc formed by the meniscus (b) electric field distribution in the Hertz deformed region and meniscus region

Thus, the following can be obtained:

$$C' = k(C_1 + C_2) + (1-k)(C_{tg1} + C_{tg2}) \quad (23)$$

$$C_{tg2} = f(c_0, \theta_2, \theta_{tg}, r_1, \varepsilon_{oil}) \quad (24)$$

$$C_{tg1} = f(c_0, \theta_{tg}, \pi, r_1, \varepsilon_{air}) \quad (25)$$

The arc formed by the meniscus in Fig. 8 can be expressed as follows:

$$(x - x_1)^2 + (y - y_1)^2 = r^2 \quad (26)$$

According to the geometric relationship, x_1 , y_1 and radius of the arc r can be obtained by the following implicit functions:

$$\begin{cases} -2R_1 r \cos \gamma + x_1^2 + y_1^2 = R_1^2 + r^2 \\ y_1 = r \cos \gamma - c \\ (R_1 \sin \theta_1 - x_1)^2 + (R_1 \cos \theta_1 - y_1)^2 = r^2 \end{cases} \quad (27)$$

Then the angle θ_{tg} between the tangent line and the Z axis can be calculated, while the parameter k is determined using a fitting approach from the FE model results. Totally 24640 FE simulations are conducted under different working conditions, which are summarized in Table 1.

Table 1 Simulation conditions

Variables	Min	Reference	Max
θ	1°	20°	60°
γ	5°	20°	50°
c/r	10^{-4}	0.001	0.1
ε_{oil}	ε_{air}	$10\varepsilon_{air}$	$30\varepsilon_{air}$
a/r	0	0.01	0.02

Based on the simulation results, it is found that k is the function of θ and γ , and obtained by means of curve-fitting as follows:

$$k = g(\theta, \gamma) = 0.2163\sqrt{\theta} + 0.4029\sqrt{\theta}\gamma + 0.0358 \quad (28)$$

The $R^2=0.995$ indicates a good fit. Combined with Eq. (23) and Eq.(28), the modified capacitance expression can be obtained as:

$$C' = g(\theta, \gamma) \cdot (C_1 + C_2) + (1 - g(\theta, \gamma)) \cdot (C_{tg1} + C_{tg2}) \quad (29)$$

The normalized capacitance $\bar{C}$ and e_θ (relative to the simulation results) of various models, for $c/r = 0.001$, $3^\circ < \theta_1 < 60^\circ$, $\varepsilon_{oil} = 10\varepsilon_{air}$, and $\gamma = 20^\circ$, are shown in Fig. 9. The results obtained using the method in Ref. [32], which assumed the completely immersed condition, are included for comparison. To fix the immersed condition, the capacitance model in Ref. [32] is extended to a θ_1 -dependent model here by performing a piecewise integration over the oil and air regions, and the corresponding results are also shown in Fig. 9(a) for comparison. Fig. 9(b) further compares e_θ of the extended model of Ref. [32], the proposed model before and after the contact angle correction.

It can be found in Fig. 9(a) that the original model in Ref. [32], which assumed the fully immersed condition, yields a constant $\bar{C}$ regardless of the oil reservoir angle and coincides with the extended model of Ref. [32] at around $\theta_1 = 45^\circ$. This proves the importance of considering the effect of θ_1 . In contrast, the capacitance of the proposed method closely aligns with the simulation results. As shown in Fig. 9(b), the extended model based on Ref. [32] underestimates the capacitance by 6% to 16% depending on the value of θ_1 , whereas the proposed model reduces the maximum error to below 3.9% before correction and further to below 0.3% after correction for reference cases. Across all the operating conditions listed in Table 1, the maximum error remains below 2.5%. These comparisons prove the accuracy and significance of the proposed method.

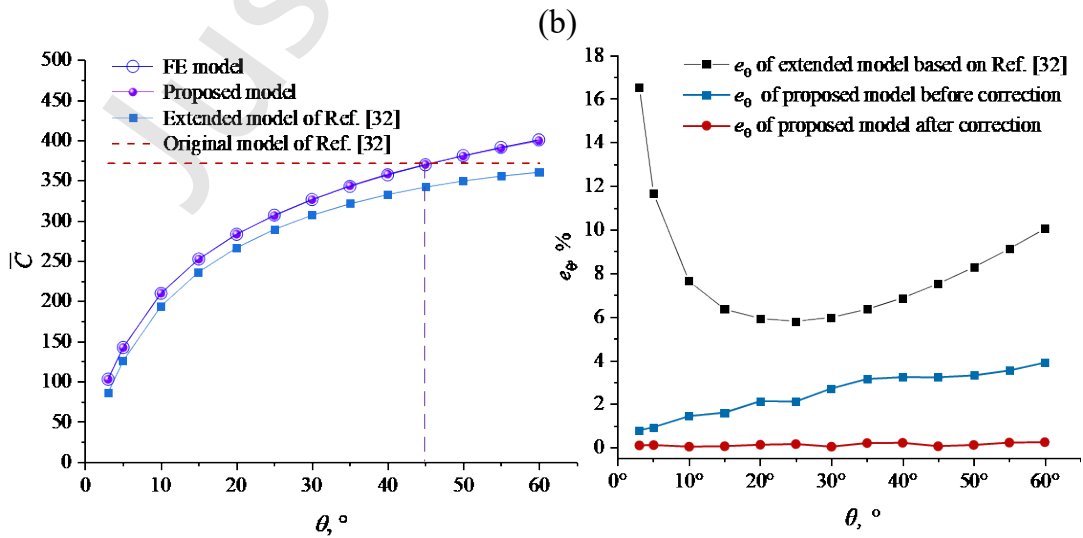


Fig. 9. Comparison of $\bar{C}$ and e_θ among various models (a) variation of $\bar{C}$ and e_θ with

Fig. 10(b) θ_0 before and after contact angle correction

2.6 Consideration of the dynamic conditions

The proposed model can also be extended to dynamic conditions by adjusting the integration limits to account for the geometry of the oil reservoir. The top view of the typical oil reservoir in EHL conditions and its cross-sectional view along the axis of symmetry at the inlet are illustrated in Fig. 10. Here, ϕ_1 denotes the outer boundary of the oil reservoir, ϕ_2 corresponds to the Hertzian deformation boundary, and ϕ_{in} defines the smallest boundary of the inlet zone. Due to symmetry, only half of the inlet ($0 \leq \phi \leq \phi_{in}$) and cavitation ($\phi_{out} \leq \phi \leq \phi_1$) regions are considered, and the boundary $\phi = \phi_1$ is treated as a function of ϕ . In the inlet region, θ is assumed as linear form θ_{in} to θ_1 , which can be expressed as:

$$\theta(\phi) = \begin{cases} (\theta_1 - \theta_{in}) / \phi_1 \cdot \phi + \theta_{in}, & \phi \in [0, \phi_1] \\ \theta_1, & \phi \in (\phi_1, \phi_{out}] \\ \theta_2, & \phi \in (\phi_{out}, \pi] \end{cases} \quad (30)$$

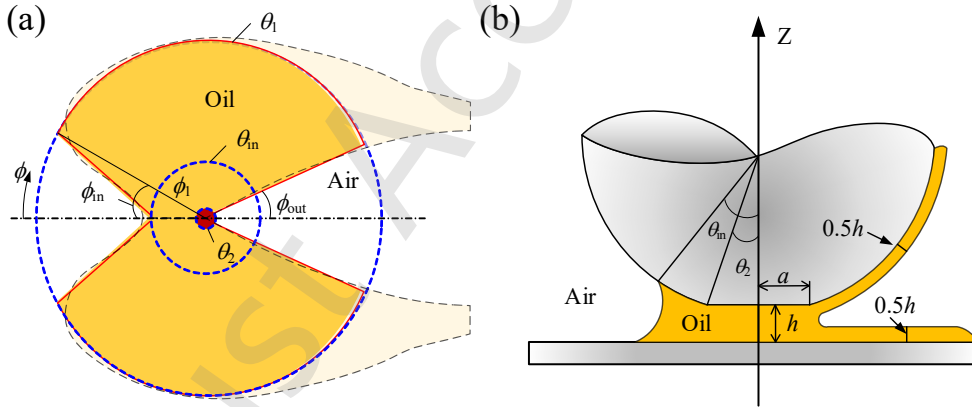


Fig. 10. Shape of oil reservoir under dynamic conditions. (a) top view of the oil reservoir (b) cross-sectional view along the axis of symmetry at the inlet

In the cavitation region, the oil film is assumed to usually attach to the ball and disc surfaces with a thickness of $0.5h$. Thus, the equivalent dielectric constant $\varepsilon(\theta)$ in the cavitation region can be approximated as:

$$\varepsilon(\theta) = \begin{cases} \varepsilon_{oil} & \theta \in [0, \theta_2] \\ ((h_{gap} - h) \cdot \varepsilon_{air} + h \cdot \varepsilon_{oil}) / h_{gap} & \theta \in [\theta_2, \pi/2] \\ \varepsilon_{air} & \theta \in [\pi/2, \pi] \end{cases} \quad (31)$$

Based on Eq. (14) and the relationship between θ and ϕ (Eq. (30)), the surface

charge over the oil reservoir region can be expressed as:

$$Q = 2\varepsilon_{\text{oil}} \int_0^\pi \int_{\theta_2}^{\theta(\phi)} \frac{1}{h_\xi} \frac{\partial \varphi}{\partial \xi} h_\eta h_\phi \bigg|_{\xi=\xi_1} d\theta d\phi \quad (32)$$

Similarly, the contributions from the air region (without cavitation) and the cavitation region can be written as:

$$Q = 2\varepsilon_{\text{air}} \int_0^\pi \int_{\theta(\phi)}^\pi \frac{1}{h_\xi} \frac{\partial \varphi}{\partial \xi} h_\eta h_\phi \bigg|_{\xi=\xi_1} d\theta d\phi \quad (33)$$

$$Q = 2\varepsilon(\theta) \int_0^{\phi_{\text{out}}} \int_{\theta_2}^\pi \frac{1}{h_\xi} \frac{\partial \varphi}{\partial \xi} h_\eta h_\phi \bigg|_{\xi=\xi_1} d\theta d\phi \quad (34)$$

Then using $C = Q/U$, the capacitance formed by the ball-disc system can be obtained.

3. Experimental implementation

3.1 Test for capacitance of specific oil reservoirs

To verify the proposed method for oil film capacitance, an experimental setup was designed, as shown in Fig. 11. A digital LCR meter is applied between the ball and disc, which are positioned inside the shield box with copper foil covered. The ball and disc are connected to the high and low potential terminals of the LCR meter, respectively, and the shield box is grounded. To accurately control the shape of the oil reservoir, a high-precision ring fabricated using 3D printing (microArch®S240) is placed between them as the wall of the oil reservoir (Fig. 11(b)). The geometry of the ring is designed to generate a cavity zone the same as the specific oil reservoir when ball is placed on it. The ball is cut to create a flat circular surface to simulate the Hertz deformed region. The overall experimental device is shown in Fig. 11(c), while the ball-ring-disc system with various rings is shown in Fig. 11(d). First, the oil is added to fill the entire ring, and then the ball is lowered until fully in contact with the ring, after which the measurements begin. The capacitance value of the ball-ring-disc system can be obtained by the LCR meter.

The theoretical capacitance should involve the contribution from the ring, denoted as C_{ring} . Thus, the total capacitance can be divided into four parts: C_1 , C_{ring} , C_2 and C_3 , while the former three parts can be calculated using corresponding dielectric constants

and modifying the integral intervals in Eq. (15) to $(\theta_3, \square\square)$, (θ_1, θ_3) , (θ_2, θ_1) , respectively. θ_3 is the angle for the outer boundary of the ring, as shown in Fig. 11(b).

The test ball, made of GCr15 bearing steel, has a radius of 12.8 mm, and the disc is much larger than the test ball. Three rings with different heights (relative dielectric constant 3.45) are printed to create a distance c between the ball and disc as $c/r=0.0018$, 0.01 and 0.1 for the cases of undeformed ball. For all the rings, the oil reservoir angle $\square_1$ and contact angle $\square$ (see Fig. 1(b)) are both set as 20° , which are controlled by the ring geometry. Using the first ring with $c/r = 0.0018$, the radii of Hertz deformed region are set to $a/r = 0.04$, 0.08 and 0.16 for deformed ball cases, while the resultant oil film thickness h can be calculated using Eq. (17).

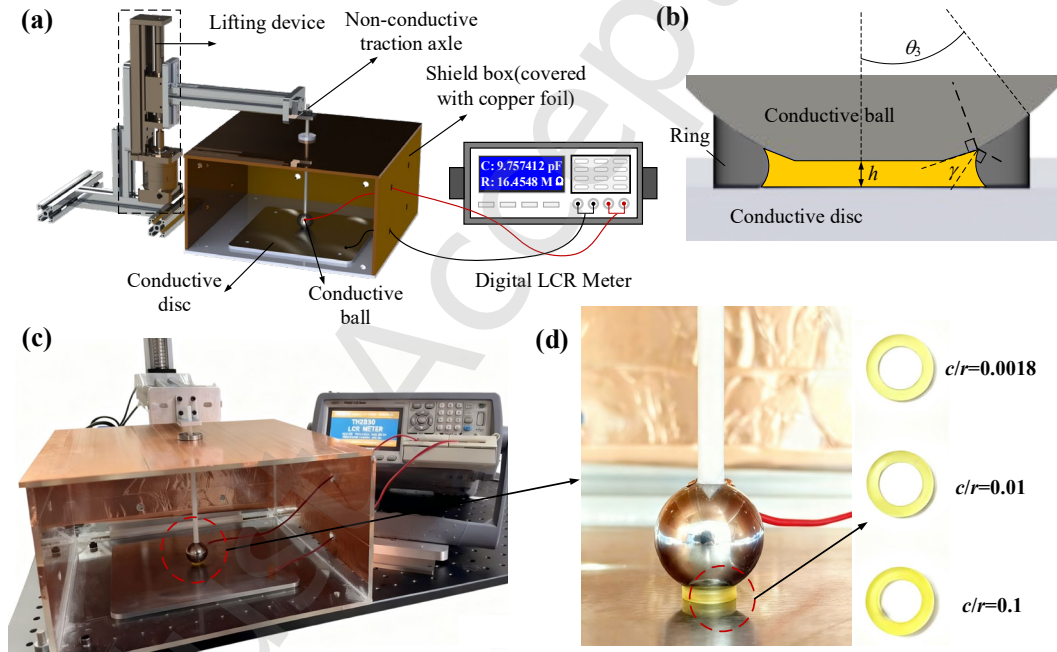


Fig. 11 Ball-disc capacitance experiment (a) schematic diagram of the ball-disc test rig (b) the ball-disc contact area (c) photo of the ball-disc test rig (d) ball-ring-disc system and rings

The roughness of the sphere surface and the cut surface representing the Hertz deformed region were measured using the CHOTEST WX100 white light interferometer from Chotest Technology Inc., with typical roughness values of $R_a = 0.05 \mu\text{m}$ and $0.64 \mu\text{m}$, respectively, as shown in Fig. 12 for their surface morphologies. Thus, the clearance between the ball and disc is much larger, with values of $23 \mu\text{m}$ for

$c/r = 0.0018$, than the roughness. Two types of test oils in this experiment are used: PAO-8 (Poly Alpha Olefin 8) and PEG-200 (Polyethylene Glycol 200), with relative dielectric constants 2.12 and 19.2, respectively. These values remain nearly constant over a frequency range of 100 kHz to 1 MHz [40]. All the tests were conducted under ambient temperature conditions. An AC sinusoidal voltage with an amplitude of 1 V and a frequency of 100 kHz was applied through an LCR meter. Each test lasted 1 minute, with data recorded every 1 ms, and for each case the test was repeated three times.

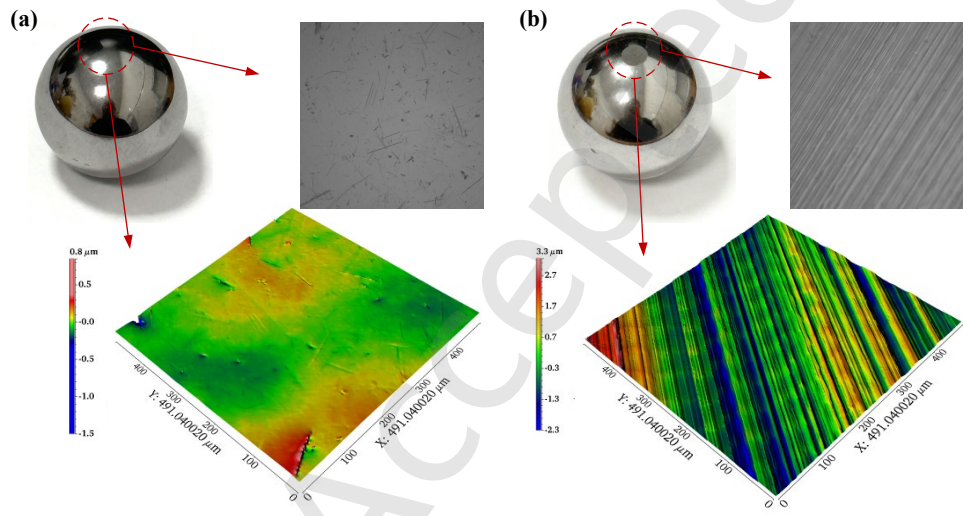


Fig. 12 The surface morphology captured by the white light interferometer (a) the sphere surface of the ball (b) the cut surface of the ball

3.2 Experimental verification of the static model

Fig. 13 shows capacitance between ball and disc under different experimental conditions. Based on Fig. 4, an advanced FE model of ball-ring-disc is built, and results from FE simulation are also shown for comparison. Fig. 13(a) and (b) show the capacitance in the cases with undeformed balls, using PAO-8 and PEG-200, respectively. Fig. 13(c) and (d) present the results for the cases considering Hertz deformed region for the two lubricants. Detailed experimental results and comparisons with the proposed method in different experimental conditions are also given in Table 2.

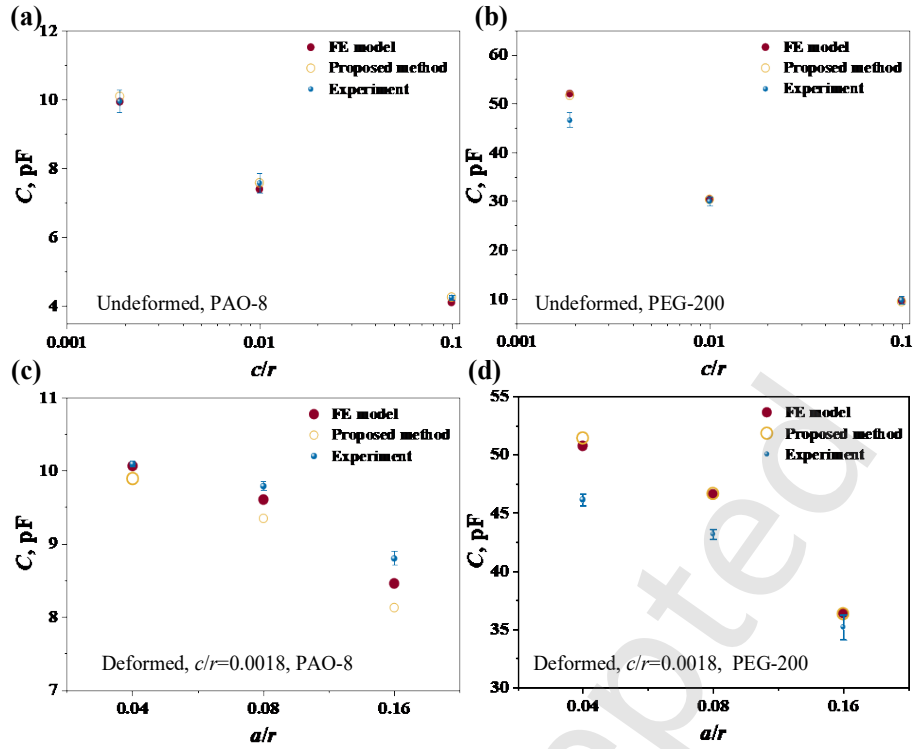


Fig. 13 Capacitance in different conditions (a) undeformed ball, PAO-8 (b) undeformed ball, PEG-200 (c) deformed ball, PAO-8 (d) deformed ball, PEG-200

Table 2 Experimental results and comparisons with the proposed method

	Parameter	Calculation (pF)	Experiment mean value (pF)	Mean error (%)	Maximum error (%)
PAO-8	$c/r=0.0018$	10.102	9.960	1.40	3.32
	$c/r=0.01$	7.586	7.579	0.09	1.08
	$c/r=0.1$	4.261	4.246	0.35	3.88
PEG-200	$c/r=0.0018$	51.748	46.646	10.9	11.2
	$c/r=0.01$	30.361	30.409	0.16	2.70
	$c/r=0.1$	9.301	9.723	4.34	5.94
PAO-8	$a/r=0.04$	10.866	10.100	7.58	8.39
	$a/r=0.08$	10.089	9.792	3.03	4.30
	$a/r=0.16$	8.339	8.803	5.27	7.23
PEG-200	$a/r=0.04$	50.735	46.124	9.34	10.71
	$a/r=0.08$	46.564	43.169	7.86	8.30
	$a/r=0.16$	37.164	35.179	5.64	11.03

Fig. 13 and Table 2 indicate that the proposed method achieves a high level of accuracy across various conditions, with mean error below 10% and maximum error below 12%. These results validate the model's reliability and applicability in practical measurements. The larger errors observed at smaller clearances may result from the limited 3D printing precision of the ring, which becomes very sensitive in these cases. Additional factors, such as the surface roughness of the cut surface and the possible slight unparallel installation with the disc surface, may also contribute to the errors.

3.3 Experimental verification of the dynamic model

Dynamic experiments were performed using a ball-disc test rig adapted from Ref. [41]. The ITO-coated glass disc and the ball were connected to the LCR meter via a carbon brush and a conductive shaft with a conductive slip ring, respectively. A sinusoidal AC voltage of 1 V at a frequency of 100 kHz was used for the measurements. Film thickness and oil reservoir shape were measured using optical interferometry and a high-speed camera, while the actual capacitance signal was obtained by the LCR meter. To minimize dripping and splashing, a high-viscosity lubricant (PAO-20, $\varepsilon \approx 2.2$) was used. The ball rotation speeds of 50, 100, 150, 200, 250, and 300 mm/s were tested under sliding conditions. Each test lasted 1 minute, with data recorded every 1 ms, and for each case the test was repeated two times.

Steady-state images of the oil reservoir were captured at the same microscope magnification, as shown in Fig. 14, where the oil reservoir boundaries, ball sliding direction, and measurement locations are indicated. It can be observed that a butterfly-shaped oil reservoir is formed [42]. With increasing speed, the maximum radius increases while the inlet-center distance decreases. The pixel-to-length relationship was calibrated using a 1 mm scale. To determine φ_{in} and φ_{out} , the curved oil reservoir contours at the inlet and outlet were approximated by equivalent straight lines, following the approach adopted in Ref. [20]. φ_{in} and φ_1 were calculated as the arcsines of the measured inlet length and mean oil reservoir radius divided by the ball radius, respectively. The central film thicknesses and the oil reservoir shape parameters are summarized in Table 3.

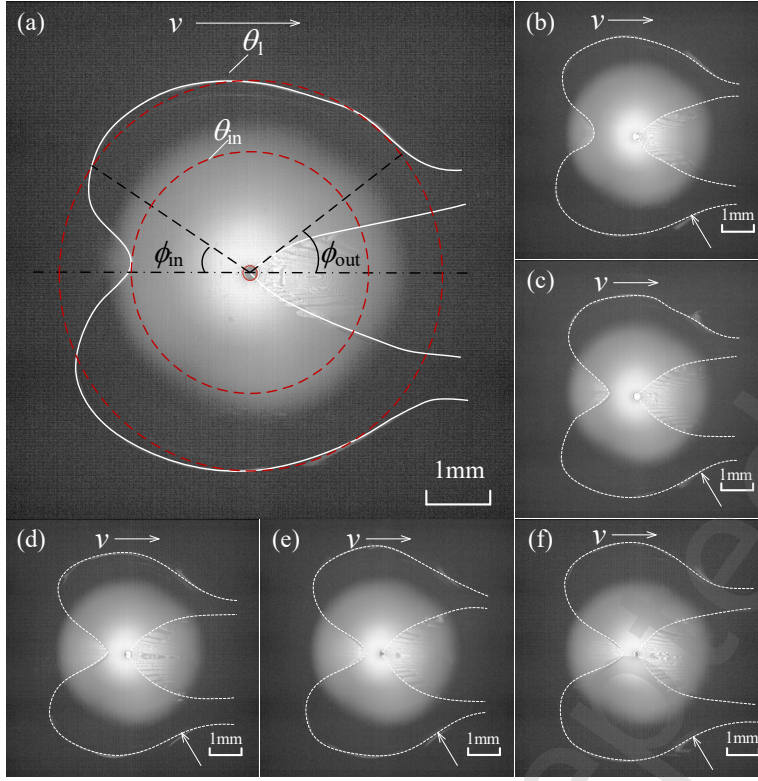


Fig. 14 Images of the oil reservoir in tested cases (a) $v=50\text{mm/s}$ (b) $v=100\text{mm/s}$ (c) $v=150\text{mm/s}$ (d) $v=200\text{mm/s}$ (e) $v=250\text{mm/s}$ (f) $v=300\text{mm/s}$

Table 3 The central film thickness and the shape parameter of oil reservoir

Speed	a	φ_{in}	φ_1	φ_{in}	φ_{out}	h
(mm/s)	(μm)	($^{\circ}$)	($^{\circ}$)	($^{\circ}$)	($^{\circ}$)	(nm)
50	110.43	7.88	12.82	25.78	36.87	77.4
100	118.10	5.36	13.45	29.25	41.12	143.0
150	118.41	3.54	13.60	32.80	48.78	192.4
200	117.76	2.65	13.84	35.62	50.45	231.7
250	114.00	2.21	14.30	38.43	51.61	258.8
300	111.35	1.79	14.95	39.42	53.75	302.0

The measured capacitance, the predictions of the dynamic model, and their relative errors are shown in Fig. 15. Fig. 15(a) presents violin plots of capacitance at different rotational speeds, including the data distribution, mean values (square symbols), and median values (circle symbols). Fig. 15(b) compares the experimental and predicted values together with their errors. Fig. 15(b) shows that the maximum error ranges from 3% to 13% across lubrication conditions from fully flooded to nearly starved. Overall,

the model slightly underestimates the measured capacitance, which may be attributed to the presence of a minimum film thickness within the Hertz contact region [43], leading to higher actual capacitance.

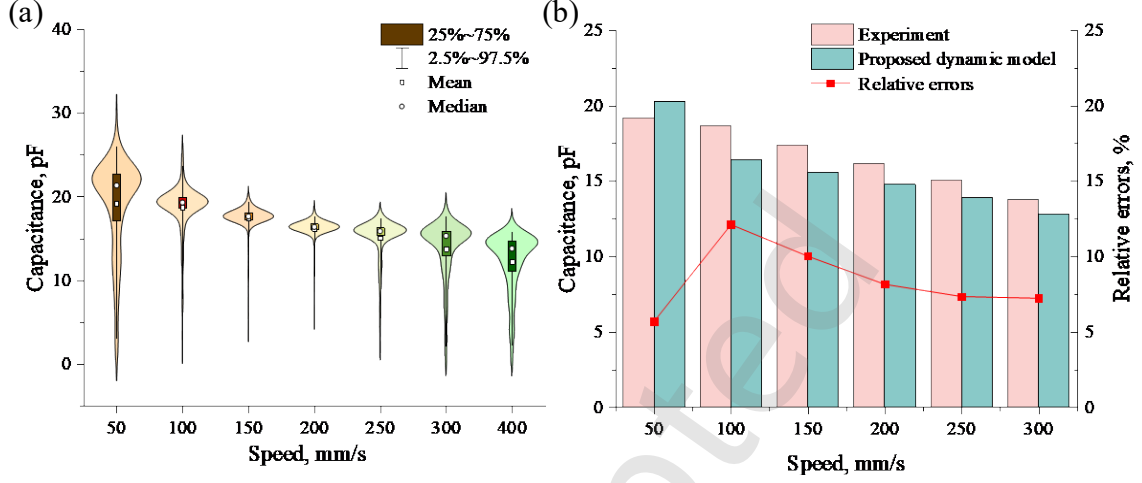


Fig. 15 Measured vs. calculated capacitance and relative errors (a) the capacitance data at different rotation speeds (b) comparison of measured and calculated capacitance with errors

4. Results and discussion

This section investigates the effect of different parameter conditions on capacitance, including minimum clearance c , oil reservoir angle φ_0 , contact angle γ , and the Hertz contact radius a .

4.1 Effect of minimum clearance c

To make the results universal, c/r is used as the variable for cases of undeformed ball and the selected values of c/r span three orders of magnitude in this section. The parameters are set as $10^{-4} \leq c/r \leq 10^{-1}$, $\gamma = 20^\circ$, and $\varepsilon_{oil} = 10\varepsilon_{air}$, for different φ_0 ($\varphi_0 = 5^\circ, 10^\circ, 20^\circ, 40^\circ, 60^\circ$). Fig. 16 illustrates the $\bar{C}$ at different c/r . It can be observed that $\bar{C}$ decreases with increasing c/r for all the φ_0 values. For given φ_0 , as c/r decreases, the capacitance increases significantly, which is because the spatial distribution of surface charge density basically remains unchanged, while its amplitude increases significantly. For given c/r , the capacitance rises as φ_0 increases.

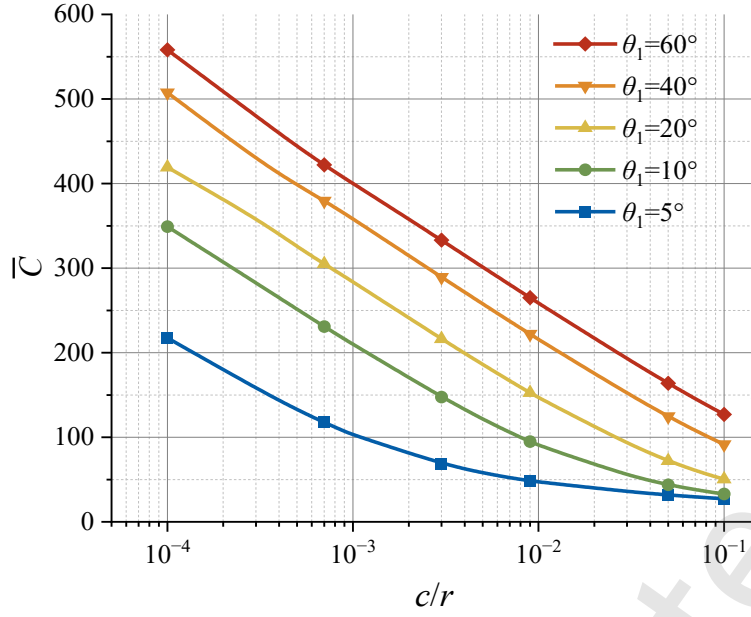


Fig. 16. $\bar{C}$ at different c/r

4.2 Effect of oil reservoir angle θ_1

The variations in the oil reservoir angle, which reflects changes in the amount of lubricant, play a critical role in determining the capacitance characteristics and representing different lubrication conditions. The effect of θ_1 on capacitance is shown in Fig. 17(a). The range of θ_1 is set as $1^\circ \leq \theta_1 \leq 60^\circ$ for different c/r ($c/r = 10^{-4}, 5 \times 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$), and all other conditions are the same as in Sec. 4.1. To observe the trend of change, the second derivative Δ of capacitance with respect to θ_1 is also presented. This derivative reflects the degree of linearity between capacitance and θ_1 under different values of c/r . The results are shown in Fig. 17(b).

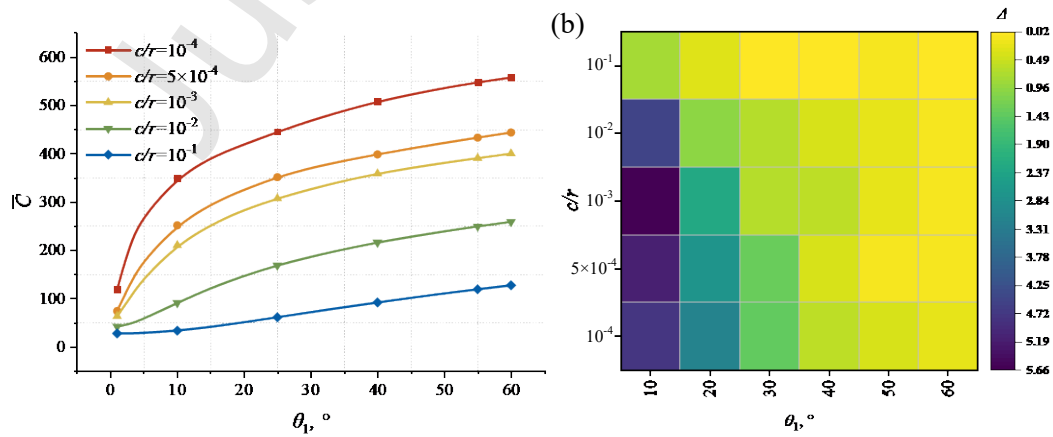


Fig. 17. $\bar{C}$ and Δ at different θ_1 (a) $\bar{C}$ at different θ_1 (b) Δ at different θ_1 and c/r

From Fig. 17(a), it can be found that for a given c/r , the capacitance increases with

φ_0 . For an angle less than around 10° , the capacitance increases rapidly, beyond which the rate of increase slows down. This tendency becomes less pronounced as c/r increases. It can be seen from Fig. 17(b) that the relationship between $\bar{C}$ and φ_0 exhibits nearly linear when both c/r and φ_0 are large, whereas it becomes non-linear when both are small. Because at the smaller φ_0 , the boundary of the oil reservoir region is close to the ball tip (the minimum clearance position). Thus, as θ increases the area filled with oil between two conductors increases much faster than the gap distance, resulting in fast increase of $\bar{C}$. At larger φ_0 , as φ_0 increases, the growth of area and gap distance becomes similar, resulting in more linear change of $\bar{C}$. Therefore, combined with the discussion in Sec. 4.2, it shows that the effect of the oil reservoir angle φ_0 on capacitance should be considered in the conditions where requires accurate capacitance calculation, especially under the condition of low oil film thickness and smaller φ_0 . In the case of large c/r and full oil immersion, the influence of φ_0 on capacitance can be estimated approximately according to the linear relationship.

4.3 Influence of contact angle γ

In this section, $\bar{C}$ for different contact angle φ are investigated with $\varepsilon_{\text{oil}} = 10\varepsilon_{\text{air}}$, $a = 0$ and c/r is 10^{-4} , 5×10^{-4} , 10^{-3} , 10^{-2} and 10^{-1} . The relative variation d_γ is introduced to show the effect of γ and defined as:

$$d_\gamma = \frac{\bar{C}_{\gamma=50^\circ} - \bar{C}_{\gamma=10^\circ}}{\bar{C}_{\gamma=50^\circ}} \times 100\% \quad (35)$$

The two cases of $\varphi = 10^\circ$ and 50° represent the lower and upper limits, respectively, of contact angle of common lubricants used in engineering. Thus, d_γ reflects the effect of the maximum change in φ on the capacitance values. As shown in Fig. 18, for c/r less than 10^{-3} , the effect of φ on capacitance is within 4% for all the values of φ_0 . For c/r larger than 10^{-2} , the effect becomes more significant, reaching up to 10% for larger c/r . In fact, the effect of contact angle on electric field and surface charge density is local and exists only in the meniscus region (see Fig. 8), which can be revealed from the FE model results. Therefore, the effect of φ on the value and trend of overall electric field and surface charge density is small. In addition, for c/r less than 10^{-3} , the rate of change is approximately linear with φ_0 . However, for c/r larger than 10^{-3} , a significant

nonlinear variation is observed at smaller β , while it exhibits a smooth linear increase at larger β .

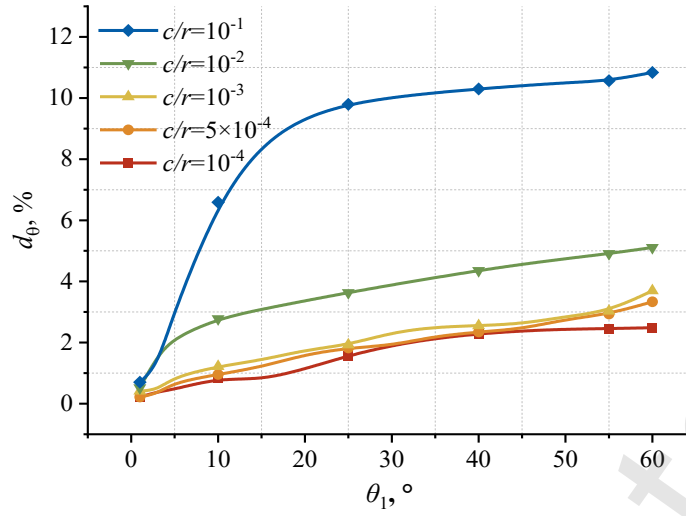


Fig. 18. Relative variation d_γ at different θ_1 and c/r

Based on the above discussion, it can be concluded that the effect of the contact angle on the capacitance value can be almost neglected when c/r is less than 10^{-2} , while its effect should be considered when θ_1 and c/r are large.

4.4 Influence of Hertz contact radius a and film thickness h

The change in Hertz contact radius a can significantly affect the capacitance. In this section, the elastic deformation caused by external load is considered. The effect of a/r on total capacitance $\bar{C}$ and the percentage of external capacitance (contributed by the oil reservoir and air region) are shown in Fig. 19(a) and Fig. 19(b), respectively, for $\epsilon_{oil} = 10\epsilon_{air}$, $\beta = 20^\circ$ and $\theta_1 = 20^\circ$. The effect of film thickness h/r are shown in Fig. 19 (c) and (d), respectively, for $\beta = 3^\circ$ and $\beta = 60^\circ$, representing the conditions of oil starvation and fully-flooded lubrication.

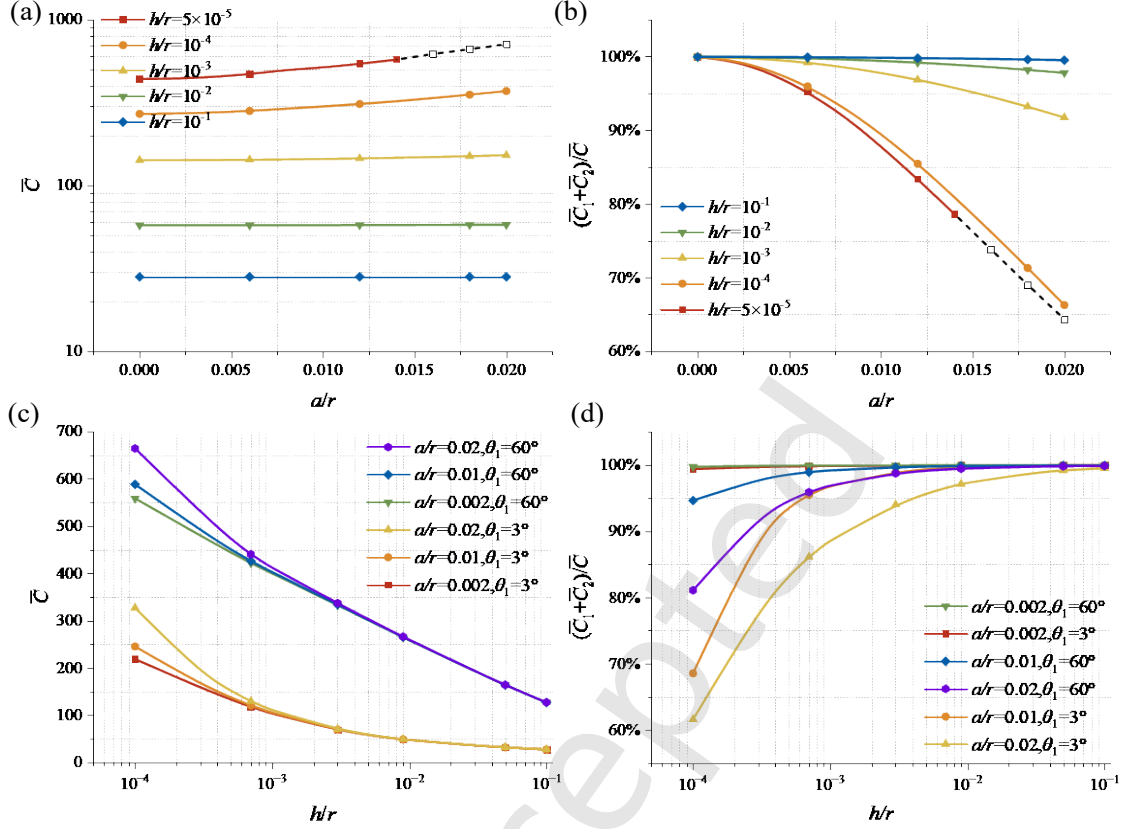


Fig. 19. Change of $\bar{C}$ and $(\bar{C}_1 + \bar{C}_2)/\bar{C}$ in different a/r and h/r (a) $\bar{C}$ vs. a/r for different h/r (b) $(\bar{C}_1 + \bar{C}_2)/\bar{C}$ vs. a/r for different h/r (c) $\bar{C}$ vs. h/r for different a/r and θ_1 (d) $(\bar{C}_1 + \bar{C}_2)/\bar{C}$ vs. h/r for different a/r and θ_1

It can be seen from Fig. 19(a) that $\bar{C}$ increases with increasing a/r , though the increase is less pronounced at large h/r and more significant at small h/r . Fig. 19(b) illustrates that at a small a/r , the external capacitance $(\bar{C}_1 + \bar{C}_2)$ contributes almost 100%, regardless of the oil film thickness, which is reasonable for the case of small Hertz deformation. As a/r increases, the contribution of external capacitance decreases, and this decrease becomes more pronounced for larger a/r , as the capacitance contributed by the Hertz deformed region with h/r increases faster than the external area, leading to the rapid decreasing of contribution by the external capacitance when a/r increases.

It can be found in Fig. 19(c) that for a fixed θ_1 and a/r , $\bar{C}$ decreases with a decreasing rate as h/r increases. This is because at small h/r , tiny changes in distance significantly affect the total charge, causing the capacitance to change rapidly. As h/r

increases, capacitance contributed by external area decreases more slowly than Hertz deformed region, making it decrease more slowly and gradually level off. From Fig. 19(d), it can be found that when h/r is large, the external capacitance contributes almost 100% under $0 < a/r < 0.02$, regardless of oil starvation or fully flooded lubrication. However, as h/r decreases, this contribution declines because the contribution from the oil film in Hertz deformed region increases. It should be noted that the dot line with three black points in Fig. 19(a) and (b) corresponds to extreme conditions (i.e., extremely large a with a small r and meanwhile very small h/r), where the equipotential circle of the undeformed ball may overlap with the disc, resulting in calculation involving the complex number field. For simplicity and high efficiency, the capacitance outside the Hertz deformed region is calculated by assuming that the clearance between the equipotential circle of the undeformed ball and the disc is extremely small (i.e., $h/r \leq 10^{-8}$), while that of the Hertz deformed region maintains the original calculation. This adjustment calculation is also validated by FE model.

It is observed that the contribution of external capacitance is greater than 50% under the above conditions, which indicates that it is necessary to evaluate the external capacitance, especially for cases of light load or fully flooded lubrication. It can also be found that the contribution of external capacitance is significantly dependent on the lubrication conditions, suggesting that using a fixed ratio between the total capacitance and the capacitance by Hertz deformed region result in inaccuracy in determining the film thickness.

4.5 Contribution of surrounding air

It is also important to clarify the contribution of air surrounding the contact area to the total capacitance. In this section, $\bar{C}$ at different ϵ_{oil} and c/r , for $\angle_{\square\square} = 20^\circ$, $\gamma = 20^\circ$ and $a = 0$, are calculated. The capacitance contributed by the air and oil are discussed. Fig. 20(a) shows the change of $\bar{C}$ with c/r when $\epsilon_{oil}/\epsilon_{air}$ is 2, 5 and 10, and Fig. 20(b1) -(b3) shows the contribution percentage of air in these conditions.

The capacitance with and without air is shown in Fig. 20(a) by the solid and dashed lines, respectively. It can be seen from Fig. 20(a) that, regardless of whether air is considered, the capacitance decreases as c/r increases, but the gap between them

increases, indicating higher air capacitance contributions. For a constant c/r , as ϵ_{oil} increases, $\bar{C}$ increases and the contribution of air to the total capacitance decreases progressively. The quantitative contribution of air capacitance is shown in Fig. 20(b1)-(b3). It can be found that for the larger ϵ_{oil} and smaller c/r , the air capacitance can be negligible; but for the values of ϵ_{oil} close to ϵ_{air} and larger c/r , the air capacitance may surpass that of oil. Therefore, it is important to account for the surrounding air capacitance when the capacitance of EHD contact area is evaluated under low ϵ_{oil} and large c/r .

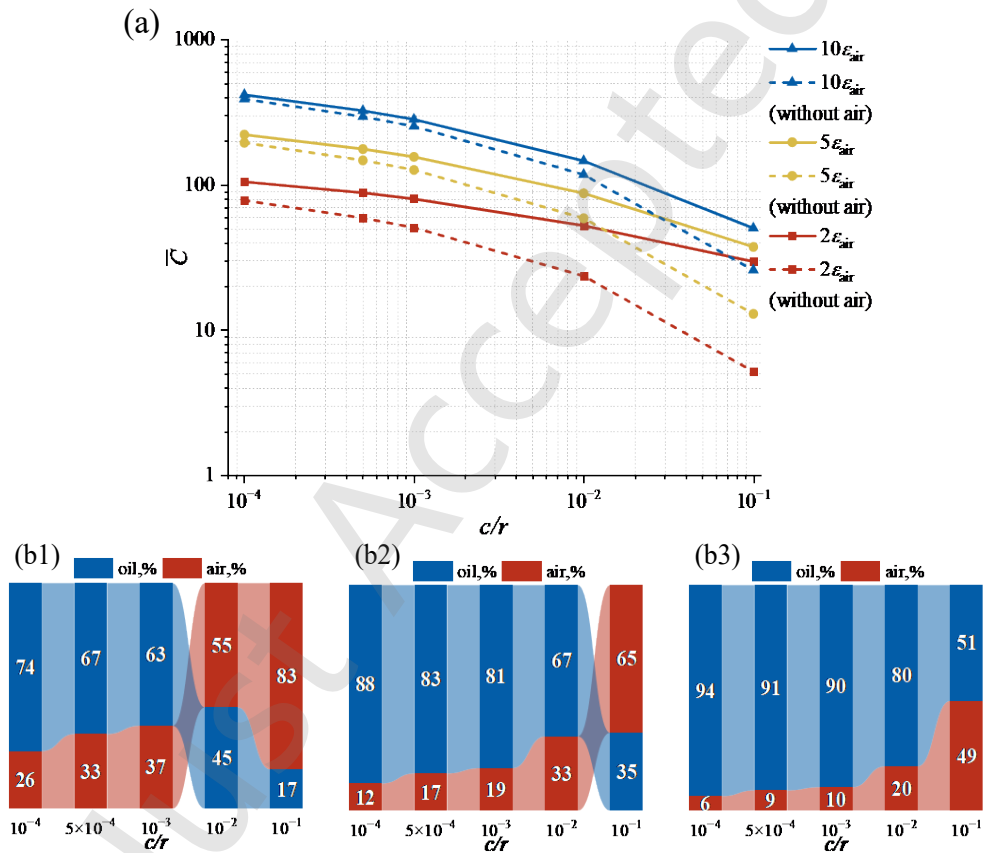


Fig. 20. $\bar{C}$ and contributions of air and oil in different c/r and ϵ_{oil} (a) $\bar{C}$ in different c/r and ϵ_{oil} (b) contributions of air to the total capacitance (b1) $\epsilon_{oil}=2\epsilon_{air}$ (b2) $\epsilon_{oil}=5\epsilon_{air}$ (b3) $\epsilon_{oil}=10\epsilon_{air}$

Conclusion

In this study, an analytical method is proposed for solving the capacitance and electrical impedance for lubricated ball-disc contact problem, enabling the consideration of more realistic factors in engineering. The method proposed in this

paper is verified with FE model simulations and experiment. A correction formula obtained by fitting is used to further consider the effect of contact angle.

The effects of minimum clearance c (or film thickness h), oil reservoir angle φ , contact angle γ , Hertz contact radius a and contribution of surrounding air on the capacitance $\bar{C}$ are systematically investigated. As c/r (or h/r) increases, $\bar{C}$ decreases, regardless of occurrence of ball deformation. As θ_1 increases, $\bar{C}$ increases, corresponding to higher external capacitance contribution, and $\bar{C}$ shows a rapid change at small φ and c/r (or h/r), indicating that the oil reservoir angle should be considered for an accurate solution. The contact angle on the capacitance can be almost neglected for c/r (or h/r) less than 10^{-2} . As a/r increases, $\bar{C}$ increases, and the contribution of external capacitance decreases for larger a or smaller c/r (or h/r). When ε_{oil} close to ε_{air} and larger c/r (or h/r), the surrounding air capacitance makes a significant contribution.

As an analytical solution, the method proposed in the present work can be used as the benchmark of capacitance solution using other methods. Future work for the present method could be considering the non-axisymmetric oil immersion for dynamic rolling conditions and applying it to the whole bearing model.

Acknowledgements

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Nomenclature

a	Hertz contact radius
C	Capacitance
$\bar{C}$	Normalization capacitance

c	Clearance of ball-disc
c_0	Ball disc clearance after deformation
$d_{\square}$	Maximum effect of change on γ for the capacitance values
e_{θ}	Percentage error between simulation and calculation
h	Oil film thickness
m	Breakdown ratio
Q	Surface charge on sphere
R_{10}	Resistance for total breakdown
R_1	Resistance in the breakdown region
r_1, r_2	Radius of inner and outer ring
U	Electric potential difference
V	Voltage on the ball
Z_1	Impedance
<i>Greek letters</i>	
θ_1	Oil reservoir angle
φ	Potential between the ball and disc
$\square$	Contact angle of oil
ε	Dielectric constant

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