

T-Fuzzy Subhypernear-Modules

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ABSTRACT

The objective of this study is to investigate the fuzzy subhypernear-modules over hypernear-ring by using a triangular norm, which is T -fuzzy subhypernear-modules are a generalization of fuzzy submodules of hypernear-modules, and some related properties are discussed. Idempotent T -fuzzy subhypernear-module of hypernear-module and subhypernear-module of hypernear-module are suggested. The relation between a level subset of T -fuzzy subhypernear-module of hypernear-module and subhypernear-modules are presented. Also, the image and inverse image of T -fuzzy subhypernear-modules under a mapping are introduced. The direct product of T -fuzzy subhypernear-modules are investigated and the finite direct product of hypernear-modules are characterized in terms of T -fuzzy subhypernear-modules.

KEYWORDS

hypernear-module; t -norm; fuzzy subhypernear-module; T -fuzzy subhypernear-module

The implementation of mathematics in another scientific field, such as computer science, plays a crucial role and represents one of the goals of the worldwide survey of masters of hyperstructure theory in recent decades Mathematician Marty^[1] proposed at the 8th Congress of Scandinavian Mathematicians. After that, a large number of works and several books were written on the subject. Currently, hyperstructures has numerous applications in different areas of mathematics and computer science^[2–5] and they are processed in countless countries around the world. This theory was improved by numerous researches^[2, 6–8]. Essential definitions and theorems about the hyperstructures can be seen in Refs. [1, 2, 5, 6, 9]. Krasner^[10] applied the notion of hyperfields, hyperrings, and then the relevant researchers^[5, 11–17] were carried out.

Hyperrings are basically rings with generally changed axioms. There are various concepts of hyperrings $(R, +, \cdot)$. If addition “+” is a hyperoperation and multiplication “ $\cdot$ ” is a binary operation, then the hyperring is called a Krasner (additive) hyperring^[10]. Rota^[18] represented a multiplicative hyperring, where “+” is a binary operation and multiplication “ $\cdot$ ” is a hyperoperation. De Salvo^[19] studied hyperrings where the additions and multiplications were hyperoperations. These hyperrings were also classified by Barghi^[20] and by Asokkumar and Velrajan^[21]. Davvaz and Leoreanu-Fotea^[9] published a book in 2007 entitled *Hyperring theory and applications*.

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After the introduction of fuzzy sets by Zadeh^[22], it has thrown the spotlight on diverse applications in the field of mathematics and related topics. This provides researchers with reasonable grounds to explore different terms and demonstrate the field of abstract algebra in a broader context of fuzzy settings. The study of fuzzy algebra structures started with the introduction of fuzzy subgroups by Rosenfeld^[23]. Checking fuzzy hyperstructures is an attractive search topic for fuzzy sets. There is a considerable body of research on the connections between fuzzy sets and hyperstructures. Davvaz^[24] proposed the concept of fuzzy hyperideals in a semihypergroup. In 2009, Davvaz^[25] focused on the concept of fuzzy hyperideals in ternary semihyperrings. The relationships between the fuzzy sets and algebraic hyperstructures have been studied by many other researchers. Everyone can continue to find fuzzy hyperstructures in Ref. [26]. The fuzzy hypernear-ring notion is defined and explored in Ref. [27]. The fuzzy hypernear-module concept is defined and edited in Ref. [28]. For more information on fuzzy algebraic structures, see Refs. [29–33].

However, Schweizer and Sklar^[34] studied the concept of the triangular norm (t -norm) and the triangular conorm (t -conorm) with the goal of generalizing the ordinary triangle inequality in a metric space to the more general probability theory in metric space. Anthony and Sherwood^[35] perspectively approached re-approached Rosenfeld's^[23] notion of fuzzy groups by using the t -norm, which has played an important role in fuzzy algebra theory. t -norm T and t -conorm S are the functions that map the unit square to the unit interval in the application. t -norm is commonly practiced to model the logical connective: conjunction in fuzzy set theory. In many areas of mathematics and artificial intelligence^[36], there are a variety of areas of triangular norms. By using a triangular norm T , the notion of T -fuzzy hyperideals in hypernear-rings has been introduced and studied by Zhan^[37].

There are several motivations for the study of T -fuzzy subhypernear-modules over hypernear-rings:

Generalization of classical algebraic structures. T -fuzzy subhypernear-modules are a generalization of classical submodules of modules over rings. Hypernear-rings are also a generalization of classical rings, so the study of T -fuzzy subhypernear-modules over hypernear-rings can be seen as a further generalization of classical algebraic structures.

Applications in fuzzy logic. Fuzzy logic is a mathematical tool for dealing with uncertainty and imprecision. The study of T -fuzzy subhypernear-modules over hypernear-rings provides a framework for applying fuzzy logic techniques to algebraic structures. This can lead to new insights and results in both algebra and fuzzy logic.

Flexible treatment of submodules. T -fuzzy subhypernear-modules provide a more flexible and nuanced treatment of submodules in the context of hypernear-rings. This allows for a more precise characterization of the structure of hypernear-rings, and can lead to new techniques for analyzing and manipulating hypernear-rings.

Triangular norm is a popular choice in fuzzy logic. Triangular norm is a popular choice in fuzzy logic, due to the simplicity and tractability. The study of fuzzy subhypernear modules with triangular norm can lead to the development of new techniques and results that take advantage of the properties of triangular norms.

Overall, the study of T -fuzzy subhypernear-modules over hypernear-rings is motivated by a desire to extend classical algebraic structures, apply fuzzy logic techniques to algebraic structures, and provide a more flexible treatment of submodules.

This theory can be used to extend other structures in a wide range of applications. The potential applications of fuzzy hyperalgebraic sets in various areas, the study of fuzzy subhypernear-modules with triangular norms have potential applications in areas such as graph theory, lattice theory, and semigroups while developing more ideas. Although, despite the existing work on fuzzy subhypernear-modules, to our knowledge there is no study in the literature of T -fuzzy subhypernear-modules of hypernear-modules. Hence, to elude this gap in the literature, in this research, we put forth a novel study of fuzzy subhypernear-modules of hypernear modules with triangular norms. The major contributions of this study are displayed as follows.

- (1) The notion of fuzzy subhypernear-modules over hypernear ring with triangular norms is presented and their algebraic properties are given.
- (2) A new generalization of the fuzzy subhypernear-modules of hypernear-modules is established and some theorems of them are proposed.
- (3) The notion of idempotent T -fuzzy subhypernear-module of hypernear-module and subhypernear-module of hypernear-module is compared, and various properties of them are investigated, and the relationship between them is given.
- (4) The relation between a level subset of T -fuzzy subhypernear-module of hypernear-module and subhypernear-module is presented.
- (5) A novel image and inverse image of fuzzy subhypernear-modules over hypernear ring with triangular norms are presented.
- (6) The direct product of T -fuzzy subhypernear-modules are suggested.
- (7) The finite direct product of hypernear-modules is characterized in terms of T -fuzzy subhypernear-modules over hypernear-ring.

The goal of this paper is to extend the fuzzy subhypernear-modules to fuzzy subhypernear-modules over hypernear-ring by using triangular norms and some related properties are discussed and thereby establish a new generalization of the fuzzy subhypernear-modules of hypernear-modules. Idempotent T -fuzzy subhypernear-module of hypernear-module and subhypernear-module of hypernear-module are compared and the relationship between them is given. The image and inverse image of T -fuzzy subhypernear-modules under a mapping are examined. The direct product of T -fuzzy subhypernear-modules are investigated and the finite direct product of hypernear-modules is characterized in terms of T -fuzzy subhypernear-modules.

1 Preliminary

We recall first the basic terms and definitions.

In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

A map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called hyperoperation or join operation on the set H , where H is a non-empty set and $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ denotes the set of all nonempty subsets of H . A hyperstructure is called the pair $(H, \circ)$ where $\circ$ is a hyperoperation on the set H . A hyperstructure $(H, \circ)$ is called a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v \quad (1)$$

If $x \in H$ and A, B are nonempty subsets of H , then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \circ B^{[2]}.$$

A non-empty subset B of a semihypergroup H , is called a sub-semihypergroup of H if $B \circ B \subseteq B$ and H is called in this case super-semihypergroup of B . Let $(H, \circ)$ be a semihypergroup. Then H is called a hypergroup if it satisfies the reproduction axiom, for all $a \in H$, $a \circ H = H \circ a = H$. An element e in a semihypergroup H is called identity if $\forall x \in H$, $x \circ e = e \circ x = \{x\}$.

An element 0 in a semihypergroup H is called zero element if $\forall x \in H$, $x \circ 0 = 0 \circ x = \{0\}$ ^[9].

A non-empty set H with a hyperoperation $+$ is said to be a canonical hypergroup if the following conditions hold

1. For every $x, y \in H$, $x + y = y + x$;
2. For every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;
3. There exists $0 \in H$, (called neutral element of H) such that $0 + x = \{x\} = x + 0$ for all $x \in H$;
4. For every $x \in H$, there exists a unique element denoted by $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$;
5. For every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$ ^[4].

A comprehensive review of the theory of hypergroups appears in Ref. [6]. For any subset A of a canonical hypergroup H , $-A$ denotes the set $\{-a : a \in A\}$. A non-empty subset N of a canonical hypergroup of H is called a subcanonical hypergroup of H if N is a canonical hypergroup under the same hyperoperation as that of H . Equivalently, for every $x, y \in N$, $x - y \subseteq N$. In particular, for any $x \in N$, $x - x \subseteq N$. Since $0 \in x - x$, it follows that $0 \in N$ ^[9].

Definition 1.1 A hypernear-ring is a triple $(R, +, \cdot)$, where R is a non-empty set with a hyperaddition “+” and a hypermultiplication “ $\cdot$ ” satisfying the following axioms:

1. $(R, +)$ is a canonical hypergroup;
2. $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, (i.e., 0 is a bilaterally absorbing element);
3. The hypermultiplication “ $\cdot$ ” is distributive with respect to the hyperoperation “+” on the left side. i.e., for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ ^[38].

Let $(R, +, \cdot)$ be a hypernear-ring. A non-empty subset R' of R is called a subhypernear-ring of $(R, +, \cdot)$ if $(R', +, \cdot)$ is a hypernear-ring ^[38].

Example 1.1 Let $R = \{x, y\}$ be a set with two hyperoperations defined as follows:

$+$	x	y	$\cdot$	x	y
x	$\{x\}$	R	x	x	x
y	R	R	y	x	R

Clearly, $(R, +, \cdot)$ is a hypernear-ring^[38].

Example 1.2 Let $R = \{0, 1, 2\}$ be a set with two hyperoperations defined as follows.

$+$	0	1	2	$\cdot$	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$	0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{1, 2\}$	R	1	$\{0\}$	R	R
2	$\{2\}$	R	$\{1, 2\}$	2	$\{0\}$	R	R

Clearly, $(R, +, \cdot)$ is a hypernear-ring^[38].

Example 1.3 Let $(R, +, \cdot)$ be a hypernear-ring. Then $(M_n(R), +, \cdot)$ is a hypernear-ring, where $M_n(R)$ is the set of all $n \times n$ matrices over R for some natural number n and the hyperoperations $+$ and $\cdot$ are defined as follows:

For $a = (a_{ij}), b = (b_{ij}) \in M_n(R)$, $a + b = \{c \in M_n(R) : c = (c_{ij}), c_{ij} \in a_{ij} + b_{ij}, 1 \leq i, j \leq n\}$ and $a \cdot b = \{c \in M_n(R) : c = (c_{ij}), c_{ij} \in \sum_{k=1}^n a_{ik} \cdot b_{kj}, 1 \leq i, j \leq n\}$ ^[38].

Let $(R, +, \cdot)$ be a hypernear-ring. A hypernear-module is a triple $(M, +, \cdot)$ over $(R, +, \cdot)$ if non-empty set M with a hyperaddition “+” and a hypermultiplication “ $\cdot$ ” holds the following conditions:

- (1) $(M, +)$ is a hypergroup (not necessarily commutative);
- (2) $\cdot : M \times R \rightarrow P^*(M)$ for all $a, b \in M$ and $r, s \in R$;
- (i) $(a + b) \cdot r = (a \cdot r) + (b \cdot r)$;
- (ii) $a \cdot (r + s) = (a \cdot r) + (a \cdot s)$;
- (iii) $a \cdot (r \cdot s) = (a \cdot r) \cdot s$;
- (iv) $a \cdot 0 = 0$ and $0 \cdot r = 0$ ^[38].

Let $(M, +, \cdot)$ be a hypernear-module. A non-empty subset of M of M is called a subhypernear-module of $(M, +, \cdot)$ if $(M', +, \cdot)$ is a hypernear-module^[28].

A mapping $\lambda : X \rightarrow [0, 1]$, where X is an arbitrary non-empty set and is called a fuzzy set in X and $\text{Im}(\lambda)$ denote the image set of λ . For $\alpha \in [0, 1]$, the set $X_\lambda^\alpha = \{x \in X | \lambda(x) \geq \alpha\}$ is called level set of λ ^[24].

In Ref. [38], Davvaz studied the notion of a fuzzy subhypernear-ring of a hypernear-ring.

A fuzzy set λ in a hypernear-ring R is called a fuzzy subhypernear-ring of R if the following statements hold: for all $a, b \in R$,

- (1) $\inf_{c \in a+b} \lambda(c) \geq \min(\lambda(a), \lambda(b))$;
- (2) $\lambda(-c) \geq \lambda(c)$;
- (3) $\lambda(ab) \geq \min(\lambda(a), \lambda(b))$.

Omran et al.^[28] introduced the concept of a fuzzy subhypernear-module of a hypernear-module over hypernear-ring.

A fuzzy set λ of a hypernear-module M over a hypernear-ring R is called a fuzzy subhypernear-module of M if the following statements hold: for all $a, b \in M$ and $c \in R$;

- 1) $\inf_{c \in a+b} \lambda(c) \geq \min(\lambda(a), \lambda(b))$;
- 2) $\lambda(-c) \geq \lambda(c)$;
- 3) $\lambda(ra) \geq \lambda(a)$.

If λ be a fuzzy subhypernear-module of M , then it is obvious that for all $a, b \in M$, $\lambda(-c) = \lambda(c)$, $\inf_{c \in a-b} \lambda(c) \geq \min(\lambda(a), \lambda(b))$.

A triangular norm (briefly, t -norm) (cf. Schweizer and Sklar^[34]) is a function $T: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying for every $x, y, z \in [0, 1]$ the following conditions:

- (1) $T(x, y) = T(y, x)$ (commutative);
- (2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotone in the right factor);
- (3) $T(x, T(y, z)) = T(T(x, y), z)$ (associative);
- (4) $T(x, 1) = x$ (having 1 as a right identity).

These four axioms are independent in the sense that none of them can be deduced from the other three. Obviously, the function $\min$ defined on $[0, 1] \times [0, 1]$ is a t -norm is defined as $T_a(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$. Other t -norms which are frequently encountered in the study of probabilistic spaces are T_m and T_p defined by $T_m(x, y) = \max(0, x + y - 1)$, $T_p(x, y) = xy$ for every $x, y \in [0, 1]$.

Also, the other t -norms

$$T_b(x, y) = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}, \quad T_c(x, y) = \frac{xy}{2 - (x + y - xy)}, \quad T_d(x, y) = \frac{xy}{x + y - xy}.$$

2 T -Fuzzy Subhypernear-Modules of Hypernear-Modules

In this section, let M denotes a hypernear-module over a hypernear-ring R unless otherwise specified. Firstly, we define the T -fuzzy subhypernear-module of hypernear-module as follows:

Definition 2.1 Let T be a t -norm. A fuzzy set λ in M is called a T -fuzzy subhypernear-module of M if it satisfies

- (1) $\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b))$;
- (2) $\lambda(-c) \geq \lambda(c)$;
- (3) $\lambda(ra) \geq \lambda(a)$;

for all $a, b \in M$ and $r \in R$.

Example 2.1 Let $R = \{0, 1, 2\}$ be a set with hyperoperation $+$ and a binary operation $\cdot$ as follows.

$+$	0	1	2		$\cdot$	0	1	2
0	{0}	{1}	{2}	and	0	0	0	0
1	{1}	{0}	{1, 2}		1	0	1	2
2	{2}	{1, 2}	{0}		2	0	2	1

Then $(R, +, \cdot)$ is a hypernear-ring. Let $M = \{0, a, b, c\}$ with the following hyperoperations.

$+$	0	a	b	c	and	$\cdot$	0	a	b	c
0	{0}	{a}	{b}	{c}		0	0	0	0	0
a	{a}	{c}	{a, b}	{a, c}		1	0	a	b	c
b	{b}	{a, b}	{0}	{b, c}		2	0	b	0	b
c	{c}	{a, c}	{b, c}	{0}						

Then $(M, +, \cdot)$ is a hypernear-module over hypernear-ring $(R, +, \cdot)$. Define a fuzzy set $\lambda : M \rightarrow [0, 1]$ as

$$\lambda(x) = \begin{cases} 1, & x = 0, \\ 0.3, & x = a, c, \\ 0.5, & x = b. \end{cases}$$

Then λ is a $(T_b, T_c, T_d, T_m, T_p)$ -fuzzy subhypernear-module of M . Let we take a fuzzy set λ in M by

$$\lambda(x) = \begin{cases} 0.3, & x = 0, \\ 0.2, & x = a, c, \\ 0.6, & x = b. \end{cases}$$

Then, λ is a T_m -fuzzy subhypernear-module of M . But λ is not a T_p -fuzzy subhypernear-module of M and λ is not a T_a -fuzzy subhypernear-module of M , since

$$\inf_{0 \in b+b} \lambda(0) = 0.3 \not\geq 0.36 = 0.6 \times 0.6 = \lambda(b) \times \lambda(b), \text{ and}$$

$$\inf_{0 \in b+b} \lambda(0) = 0.3 \not\geq 0.6 = \min\{0.6, 0.6\} = \min\{\lambda(b), \lambda(b)\}.$$

Let λ be a fuzzy set in M by

$$\lambda(x) = \begin{cases} 0.5, & x = 0, \\ 0.3, & x = a, c, \\ 0.7, & x = b. \end{cases}$$

Then λ is a T_p -fuzzy subhypernear-module of M and λ is a T_c -fuzzy subhypernear-module of M , but λ is not a T_a -fuzzy subhypernear-module of M and λ is not a T_d -fuzzy subhypernear-module of M since

$$\inf_{0 \in b+b} \lambda(0) = 0.5 \not\geq 0.7 = \min\{0.7, 0.7\} = \min\{\lambda(b), \lambda(b)\}, \text{ and}$$

$$\inf_{0 \in b+b} \lambda(0) = 0.5 \not\geq 0.53 = \frac{0.7 \times 0.7}{0.7 + 0.7 - 0.49} = \frac{\lambda(b) \times \lambda(b)}{\lambda(b) + \lambda(b) - \lambda(b) \times \lambda(b)}.$$

Example 2.2 Let $M = \{e, a, b, c\}$ and $R = \{0, 1, 2\}$ be two non-empty sets with the following hyperoperation tables.

$+$	e	a	b	c	and	$\cdot$	e	a	b	c
e	e	a	e	c		0	e	a	b	c
a	a	{a, b}	c	{e, b}		1	c	a	b	c
b	e	c	{b, c}	b		2	e	b	c	c
c	c	{e, b}	b	c		3	e	e	c	c

Then $(M, +, \cdot)$ is a hypernear-module over hypernear-ring R in Example 2.1. Define a fuzzy set $\lambda : M \rightarrow [0, 1]$ by

$$\lambda(x) = \begin{cases} 0.5, & x = e, \\ 0.4, & x = a, \\ 0.54, & x = b, \\ 0.6, & x = c. \end{cases}$$

Then λ is a T -fuzzy subhypernear-module of M .

Example 2.3 Let $M = \{1, 2, 3\}$ and $R = \{a, b, c\}$ be two non-empty sets with the following hyperoperation table.

+	1	2	3	and	.	1	2	3
1	{1}	{2}	{3}		a	{1}	{2}	{3}
2	{2}	{3}	{2, 3}		b	{2}	{1, 2}	{1, 3}
3	{3}	{2, 3}	{3}		c	{3}	{2, 3}	{1}

Then $(M, +, \cdot)$ is a hypernear-module. Define a fuzzy set $\lambda : M \rightarrow [0, 1]$ by

$$\lambda(x) = \begin{cases} 0.75, & x = 1, \\ 0.8, & x = 2, \\ 0.9, & x = 3. \end{cases}$$

Then λ is not T -fuzzy subhypernear-module of M since $\lambda(b.2) = \lambda(1) = 0.75 \not\geq 0.8 = \lambda(2)$.

T -fuzzy subhypernear-module of hypernear-module and subhypernear-module are related as it is seen in the following theorem.

Theorem 2.1 Let N be a non-empty subset of hypernear-module M . Then N is a subhypernear-module of M if and only if χ_N is a T -fuzzy subhypernear-module of M .

Proof ($\Rightarrow$): Suppose that N is a subhypernear-module of M . If $a, b \in N$ and $r \in R$, then $a + b \subseteq N$ and $ra \subseteq N$. Thus, we have

$$\inf_{c \in a+b} \chi_N(c) = 1 = T(\chi_N(a), \chi_N(b)),$$

and

$$\inf_{c \in ra} \chi_N(c) = 1 = \chi_N(a).$$

Otherwise, we have

$$\inf_{c \in a+b} \chi_N(c) \geq 0 = T(\chi_N(a), \chi_N(b)),$$

and

$$\inf_{c \in ra} \chi_N(c) \geq 0 = \chi_N(a).$$

Let $a \in M$. If $a \in N$, then $-a \in N$ and so we get

$$\chi_N(a) = \chi_N(-a).$$

If $a \notin N$, then $\chi_N(a) = 0 \leq \chi_N(-a)$.

Thus, χ_N is a T -fuzzy subhypernear-module of hypernear-module M .

($\Leftarrow$): Admit that χ_N is a T -fuzzy subhypernear-module of M . Let $a, b \in N$. Then $\chi_N(a) = 1$ and $\chi_N(b) = 1$. Hence for any $c \in a + b$, we get

$$\chi_N(c) \geq \inf_{c \in a+b} \chi_N(c) \geq T(\chi_N(a), \chi_N(b)) = 1 \text{ and } \chi_N(c) \geq \inf_{c \in ra} \chi_N(c) \geq \chi_N(a) = 1.$$

Therefore, we have $c \in N$, it follows that $a + b \subseteq N$ and $ra \subseteq N$.

Let $a \in N$ and $\chi_N(a) = 1$, so we get $1 = \chi_N(a) \leq \chi_N(-a)$, thus $-a \in N$.

Consequently, N is a subhypernear-module of M .

Proposition 2.1 If $\{\lambda_i \mid i \in \Lambda\}$ is a family of T -fuzzy subhypernear-module of M , then $\bigcap_{i \in \Lambda} \lambda_i$ is a T -fuzzy subhypernear-module of M .

Proof Let $\{\lambda_i \mid i \in \Lambda\}$ is a family of T -fuzzy subhypernear-module of M and $a, b \in M$ and $r \in R$.

Then, we get

$$\begin{aligned} \inf_{c \in a+b} \left(\bigcap_{i \in \Lambda} \lambda_i \right) (c) &= \inf_{c \in a+b} \left(\inf_{i \in \Lambda} \lambda_i(c) \right) = \\ \inf_{i \in \Lambda} \left\{ \inf_{c \in a+b} \lambda_i(c) \right\} &\geq \\ \inf_{i \in \Lambda} \{T(\lambda_i(a)), T(\lambda_i(b))\} &\geq \\ T \left\{ \inf_{i \in \Lambda} \lambda_i(a), \inf_{i \in \Lambda} \lambda_i(b) \right\} &= \\ T \left\{ \bigcap_{i \in \Lambda} \lambda_i(a), \bigcap_{i \in \Lambda} \lambda_i(b) \right\}. \end{aligned}$$

For all $a \in M$, since $\lambda_i(a) \leq \lambda_i(-a)$ for $i \in \Lambda$, we get

$$\bigcap_{i \in \Lambda} \lambda_i(a) \leq \bigcap_{i \in \Lambda} \lambda_i(-a).$$

For every $a \in M$ and $r \in R$, we get

$$\begin{aligned} \inf_{c \in ra} \left(\bigcap_{i \in \Lambda} \lambda_i \right) (c) &= \inf_{c \in ra} \left(\inf_{i \in \Lambda} \lambda_i(c) \right) = \\ \inf_{i \in \Lambda} \left\{ \inf_{c \in ra} \lambda_i(c) \right\} &\geq \\ \inf_{i \in \Lambda} \{ \lambda_i(a) \} &= \\ \bigcap_{i \in \Lambda} \lambda_i(a). \end{aligned}$$

Thus, $\bigcap_{i \in \Lambda} \lambda_i$ is a T -fuzzy subhypernear-module of M .

The following example shows that the converse of Proposition 2.1 may not be true.

Example 2.4 Consider hypernear-module M over hypernear-ring R in Example 2.2. Define fuzzy sets $\lambda_1, \lambda_2 : M \rightarrow [0, 1]$ by

$$\lambda_1(x) = \begin{cases} 0.5, & x = e, \\ 0.7, & x = a, \\ 0.54, & x = b, \\ 0.9, & x = c, \end{cases} \quad \text{and} \quad \lambda_2(x) = \begin{cases} 0.6, & x = e, \\ 0.4, & x = a, \\ 0.8, & x = b, \\ 0.6, & x = c. \end{cases}$$

Then

$$\lambda_1 \cap \lambda_2(x) = \begin{cases} 0.5, & x = e, \\ 0.4, & x = a, \\ 0.54, & x = b, \\ 0.6, & x = c, \end{cases}$$

and $\lambda_1 \cap \lambda_2$ is a T -fuzzy subhypernear-module of M , but λ_1 and λ_2 is not a T -fuzzy subhypernear-module of M since

$$\lambda_1(3.a) = \lambda_1(e) = 0.5 \not\geq 0.7 = \lambda_1(a) \text{ and } \lambda_2(2.b) = \lambda_2(c) = 0.6 \not\geq 0.8 = \lambda_2(b).$$

Proposition 2.2 Let λ be an idempotent T -fuzzy subhypernear-module of a hypernear-module M . Then $\lambda(a) \leq \lambda(0)$ for all $a \in M$.

Proof The proof is straightforward.

Proposition 2.3 Let T be a t -norm. If λ is an idempotent T -fuzzy subhypernear-module of hypernear-module M , then the set

$$M^s = \{a \in M \mid \lambda(a) \geq \lambda(s)\}$$

is a subhypernear-module of a hypernear-module M .

Proof Let $a, b \in M^s$ and it follows that $\lambda(a) \geq \lambda(s)$ and $\lambda(b) \geq \lambda(s)$. In view of λ is a T -fuzzy subhypernear-module of M , accordingly

$$\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b)) \geq T(\lambda(a), \lambda(s)) \geq T(\lambda(s), \lambda(s)) = \lambda(s).$$

Thus, $a + b \in M^s$ it means that $a + b \in P^*(M^s)$. Let $a \in M^s$ and so we get

$$\lambda(a) \geq \lambda(s) \text{ and so } \lambda(-a) \geq \lambda(a) \geq \lambda(s).$$

Hence, we get $-x \in M^s$. Let $a \in M^s$ and $r \in R$. Then, we have

$$\lambda(ra) \geq \lambda(a) \geq \lambda(s),$$

and so $ra \in M^s$. The proof is completed.

Corollary 2.1 Let T be a t -norm. If λ is an idempotent T -fuzzy subhypernear-module of hypernear-module M , then the set

$$M^\lambda = \{a \in M \mid \lambda(a) = \lambda(0)\}$$

is a subhypernear-module of a hypernear-module M .

In Ref. [39], Abu Osman studied the property of t -norm as follows.

Lemma 2.1 Let T be a t -norm. Then,

$$T(T(a, b), T(c, d)) = T(T(a, c), T(b, d)),$$

for all $a, b, c, d \in [0, 1]$.

Proposition 2.4 Let λ_1 and λ_2 are T -fuzzy subhypernear-modules of a hypernear-module M , then $\lambda_1 \wedge \lambda_2 : M \rightarrow [0, 1]$ defined by for all $a \in M$,

$$(\lambda_1 \wedge \lambda_2)(a) = T(\lambda_1(a), \lambda_2(a))$$

is a T -fuzzy subhypernear-module of hypernear-module M .

Proof Let $a, b \in M$ and so we get

$$\begin{aligned}
\inf_{c \in a+b} \{(\lambda_1 \wedge \lambda_2)(c)\} &= \inf_{c \in a+b} \{T(\lambda_1(c), \lambda_2(c))\} \geq \\
T\left(\inf_{c \in a+b} \{\lambda_1(c)\}, \inf_{c \in a+b} \{\lambda_2(c)\}\right) &\geq \\
T(T(\lambda_1(a), \lambda_1(b)), T(\lambda_2(a), \lambda_2(b))) &= \\
T(T(\lambda_1(a), \lambda_2(a)), T(\lambda_1(b), \lambda_2(b))) &= \\
T((\lambda_1 \wedge \lambda_2)(a), (\lambda_1 \wedge \lambda_2)(b)), &
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_1 \wedge \lambda_2)(-a) &= T(\lambda_1(-a), \lambda_2(-a)) \geq \\
T(\lambda_1(a), \lambda_2(a)) &= (\lambda_1 \wedge \lambda_2)(a).
\end{aligned}$$

Besides, for $a \in M$ and $r \in R$, we get

$$\begin{aligned}
(\lambda_1 \wedge \lambda_2)(ra) &= T(\lambda_1(ra), \lambda_2(ra)) \geq \\
T(\lambda_1(a), \lambda_2(a)) &= (\lambda_1 \wedge \lambda_2)(a).
\end{aligned}$$

So, the proof is completed.

Theorem 2.2 Let T be a t -norm. If λ is an idempotent T -fuzzy subhypernear-module of hypernear-module M , then λ is a fuzzy subhypernear-module of M .

Proof Let λ be an idempotent T -fuzzy subhypernear-module of hypernear-module M . Then we get $\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b))$, for all $a, b \in M$.

Since λ is idempotent, we have

$$\begin{aligned}
\min\{\lambda(a), \lambda(b)\} &= T(\min\{\lambda(a), \lambda(b)\}, \min\{\lambda(a), \lambda(b)\}) \leq \\
T(\lambda(a), \lambda(b)) &\leq \min\{\lambda(a), \lambda(b)\}.
\end{aligned}$$

Based on these reasons, we get $T(\lambda(a), \lambda(b)) = \min\{\lambda(a), \lambda(b)\}$. Thus, for all $a, b \in M$, $\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b)) = \min\{\lambda(a), \lambda(b)\}$.

Proposition 2.5 Let N be a non-empty subset of a hypernear-module of M and let λ be fuzzy set in M given by

$$\lambda(a) := \begin{cases} k_1, & \text{if } a \in N, \\ k_2, & \text{otherwise,} \end{cases}$$

$\forall k_1, k_2 \in [0, 1]$ with $k_1 > k_2$. Then λ is an idempotent T -fuzzy subhypernear-module of M if and only if N is a subhypernear-module of M .

Proof Admit that λ is an idempotent T -fuzzy subhypernear-module of M . Let $a, b \in N$ and so

$$\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b)) = k_1 \text{ it follows that } \inf_{c \in a+b} \lambda(c) \geq k_1.$$

On that account $a + b \subseteq N$. Then, let $a \in N$ and we get

$$k_1 = \lambda(a) \leq \lambda(-a) \text{ and so } \lambda(-a) = k_1.$$

It means that $-a \in N$. Let $a \in M$ and $r \in R$, then

$$k_1 = \lambda(a) \leq \lambda(ra) \text{ and so } \lambda(ra) = k_1.$$

Thus $ra \in N$, as a result N is a subhypernear-module of M . On the contrary, assume that N is a subhypernear-module of M . Let $a, b \in M$. In case that $a \in M - N$ or $b \in M - N$, then $\lambda(a) = k_2$ and $\lambda(b) = k_2$. Hence

$$\inf_{c \in a+b} \lambda(c) \geq k_2 = \min\{\lambda(a), \lambda(b)\} = T(\lambda(a), \lambda(b)),$$

and

$$\lambda(ra) \geq k_2 = \lambda(a).$$

Suppose that $a, b \in N$ and $r \in R$ and so $a + b \subseteq N$. Thus,

$$\inf_{c \in a+b} \lambda(c) = k_1 = \min\{\lambda(a), \lambda(b)\} \geq T(\lambda(a), \lambda(b)),$$

and

$$\lambda(ra) = k_1 \geq \lambda(a).$$

By reason of $a \in N$, we acquire $-a \in N$ it follows that $\lambda(a) \leq \lambda(-a)$. Therefore, λ is a T -fuzzy subhypernear-module of M .

By using level subsets, a T -fuzzy subhypernear-module of hypernear-module M can be characterized.

Proposition 2.6 Let T be a t -norm and λ be a fuzzy set of M . In case that M_λ^s is a subhypernear-module of M for all $s \in \text{Im}(\lambda)$, then λ is a T -fuzzy subhypernear-module of M .

Proof Let $a, b \in M$. Then, $\exists s_1, s_2 \in \text{Im}(\lambda)$ such that $\lambda(a) = s_1$ and $\lambda(b) = s_2$. Admit that $s_1 \geq s_2$ without loss of generality. Then $\lambda(b) = s_1 \geq s_2$, hence $a, b \in M_\lambda^{s_1}$. We have $a + b \subseteq M_\lambda^{s_1}$ and $ra \in M_\lambda^{s_1}$ because $M_\lambda^{s_1}$ is a subhypernear-module of M . So,

$$\inf_{c \in a+b} \lambda(c) \geq s_2 = \min\{s_1, s_2\} = \min\{\lambda(a), \lambda(b)\} \geq T(\lambda(a), \lambda(b)),$$

and

$$\lambda(ra) \geq \lambda(a).$$

Let $a \in M$ with $\lambda(-a) < \lambda(a)$. Taking $a_0 = \frac{1}{2}\{\lambda(-a) + \lambda(a)\}$, then $\lambda(-a) < a_0 < \lambda(a)$, and $a \in M_\lambda^{a_0}$ though $-a \notin M_\lambda^{a_0}$. This assumption leads to a contradiction. Consequently, λ is a T -fuzzy subhypernear-module of M .

Theorem 2.3 Let T be a t -norm and λ be a fuzzy set of M with $\text{Im}(\lambda) = \{s_1, s_2, \dots, s_k\}$, where $s_j > s_i$ whenever $i > j$. Assume that there exists a chain of subhypernear-modules of M , $N_0 \subseteq N_1 \subseteq \dots \subseteq N_k = M$, such that $\lambda(N_t^*) = s_t$, where $N_t^* = N_t \setminus N_{t-1}$, $N_{t-1} = \emptyset$ for $t = 0, 1, \dots, k$. Then λ is a T -fuzzy subhypernear-module of M .

Proof Let $a, b \in M$ and $r \in R$. If $a, b \in N_t^*$, then we get $\lambda(a) = \lambda(b) = s_t$ and $a + b \subseteq N_t$. Hence we have

$$\inf_{c \in a+b} \lambda(c) \geq s_t = \min\{\lambda(a), \lambda(b)\} \geq T(\lambda(a), \lambda(b))$$

and

$$\lambda(ra) \geq \lambda(a).$$

In case that $a \in N_i^*$ and $b \in N_j^*$ for every $i \neq j$. Admit that $i \geq j$ without loss of generality. Thus, we get

$$\lambda(a) = s_i < s_j = \lambda(b), \text{ and } a + b \subseteq N_i.$$

We infer that

$$\inf_{c \in a+b} \lambda(c) \geq s_i = \min\{\lambda(a), \lambda(b)\} \geq T(\lambda(a), \lambda(b)),$$

and

$$\lambda(ra) \geq \lambda(a).$$

Let $a \in M$. Then there exists N_t such that $a \in N_t^*$ for some $t \in \{0, 1, \dots, k\}$. Hence we get $\lambda(a) = s_t = \lambda(-a)$. Consequently, λ is a T -fuzzy subhypernear-module of M .

Theorem 2.4 Let λ be a T -fuzzy subhypernear-module of M and let $s \in [0, 1]$. Then,

(1) If $s = 1$, then M_λ^s is empty or is a subhypernear-module of M .

(2) If $T = \min$, then M_λ^s is empty or is a subhypernear-module of M .

Proof (1) Suppose that $s = 1$ and $M_\lambda^s \neq \emptyset$. Then $\exists a, b \in M_\lambda^s$ such that $\lambda(a) \geq 1$ and $\lambda(b) \geq 1$. Hence, we have

$$\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b)) = T(1, 1) = 1 = s,$$

so that $c \in a + b \in M_\lambda^s$. Let $a \in M_\lambda^s$ and so

$$\lambda(-a) \geq \lambda(a) \geq 1 = s.$$

Hence, we have $-a \in M_\lambda^s$. Also, let $a \in M_\lambda^s$ and $r \in R$, so

$$\lambda(ra) \geq \lambda(a) \geq 1 = s.$$

Thus, we get $ra \in M_\lambda^s$. Hence M_λ^1 is a subhypernear-module of M .

(2) If λ is a T -fuzzy subhypernear-module of M and $M_\lambda^s \neq \emptyset$, then for any $a, b \in M_\lambda^s$ such that $\lambda(a) \geq s$ and $\lambda(b) \geq s$. Hence

$$\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b)) = \min\{\lambda(a), \lambda(b)\} = \min\{s, s\} = s,$$

which implies that $a + b \in M_\lambda^s$. Similarly, it is clear that $-a \in M_\lambda^s$ and $ra \in M_\lambda^s$ by previous condition.

This proves that M_λ^s is a subhypernear-module of M .

Theorem 2.5 Let T be a t -norm and let λ be an idempotent fuzzy set in M . If $\emptyset \neq M_\lambda^s$ is a subhypernear-module of M , then λ is an idempotent T -fuzzy subhypernear-module of M .

Proof Assume that $\emptyset \neq M_\lambda^s$ with $s \in [0, 1]$. By contraposition, suppose that M_λ^s is not a subhypernear-module of M . Then there exist

$$a_0, b_0 \in M \text{ such that } \inf_{d \in a_0+b_0} \lambda(d) < \min\{\lambda(a_0), \lambda(b_0)\},$$

Taking

$$d_0 := \frac{1}{2} \left\{ \inf_{d \in a_0 + b_0} \lambda(d) + \min \{(\lambda(a_0), \lambda(b_0))\} \right\}.$$

Then, we have

$$\inf_{d \in a_0 + b_0} \lambda(d) < d_0 < \min \{(\lambda(a_0), \lambda(b_0))\},$$

and hence $a_0, b_0 \in M_\lambda^{d_0}$ and $a_0 + b_0 \notin M_\lambda^{d_0}$. This leads to a contradiction. Therefore, we get for all $a, b \in M$

$$\inf_{c \in a+b} \lambda(c) \geq \min(\lambda(a), \lambda(b)) = T\{\lambda(a), \lambda(b)\}.$$

Also, $\lambda(a_0) > \lambda(-a_0)$ for some $a_0 \in M$. Taking

$$d_0 := \frac{1}{2} \{\lambda(a_0) + \lambda(-a_0)\}.$$

Then, we have $\lambda(a_0) > d_0 > \lambda(-a_0)$, which implies that $a_0 \in M_\lambda^{d_0}$ and $-a_0 \notin M_\lambda^{d_0}$. It is a contradiction. Finally, $\lambda(a_0) > \lambda(r_0 a_0)$ for some $a_0 \in M$ and $r_0 \in R$. Taking

$$d_0 := \frac{1}{2} \{\lambda(a_0) + \lambda(r_0 a_0)\}.$$

Then, we get $\lambda(a_0) > d_0 > \lambda(r_0 a_0)$, which implies that $a_0 \in M_\lambda^{d_0}$ and $r_0 a_0 \notin M_\lambda^{d_0}$. It is also a contradiction. Therefore, λ is an idempotent T -fuzzy subhypernear-module of M .

In Ref. [28], Omran gave the definition of homomorphism of hypernear-modules.

Let $\rho : M \rightarrow M'$ be a mapping. λ be a fuzzy subset in M and λ' be a fuzzy subset in M' . Then, λ is called ρ -invariant if $\rho(x) = \rho(y)$ implies $\lambda(x) = \lambda(y)$ for all $x, y \in M$. And the inverse image $\rho^{-1}(\lambda')$ of λ' is the fuzzy subset in M defined by $\rho^{-1}(\lambda')(x) = \lambda'(\rho(x))$ for all $x \in M$. The image $\rho(\lambda)$ of λ is the fuzzy subset in M' defined by

$$\rho(\lambda)(x') = \begin{cases} \sup_{x \in \rho^{-1}(x')} \lambda(x) & \text{if } \rho^{-1}(x') \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x' \in M'$.

Let M and M' be two hypernear-modules over a hypernear-ring R . The mapping $\rho : M \rightarrow M'$ is called a hypernear-module homomorphism if for any $r \in R$ and $x, y \in M$, we have

$$\begin{aligned} \rho(x+y) &\subseteq \rho(x) + \rho(y) \quad (\rho(x+y) = \rho(x) + \rho(y)), \\ \rho(rx) &\subseteq r\rho(x) \quad (\rho(rx) = r\rho(x)), \\ \rho(0) &= 0. \end{aligned}$$

Proposition 2.7 Let T be a t -norm and let $\rho : M \rightarrow M'$ be a homomorphism of hypernear-modules. If λ is a T -fuzzy subhypernear-module of M , then $\rho(\lambda)$ is a T -fuzzy subhypernear-module of M' .

Proof Admit that λ is a T -fuzzy subhypernear-module of M . Let $a, b \in \rho(M)$. Then, we have

$$\begin{aligned}
\text{in} \rho \{ \rho(\lambda(c)) \} &= \text{in} \rho \left\{ \sup_{c \in a+b} \lambda(x) \right\} \geq \text{in} \rho \left\{ \sup_{c \in a+b} \lambda(x) \right\} \geq \\
&\text{in} \rho \left\{ \sup_{x_1+x_2 \in \rho^-(a+b)} \lambda(x_1+x_2) \right\} \geq \\
&\text{in} \rho \left\{ \sup_{x_1 \in \rho^-(a), x_2 \in \rho^-(b)} (\lambda(x_1) + \lambda(x_2)) \right\} \geq \\
&T \left\{ \sup_{x_1 \in \rho^-(a)} \lambda(x_1), \sup_{x_2 \in \rho^-(b)} \lambda(x_2) \right\} = \\
&T \{ \rho(\lambda(a)), \rho(\lambda(b)) \}.
\end{aligned}$$

Then,

$$\begin{aligned}
\rho(\lambda(-c)) &= \sup_{x \in \rho^-(c)} \lambda(x) = \sup_{-x \in \rho^-(c)} \lambda(-x) = \sup_{x \in \rho^-(c)} \lambda(x) = \rho(\lambda(c)), \\
\rho(\lambda(rc)) &= \sup_{x \in \rho^-(rc)} \lambda(x) \geq \sup_{x_1 x_2 \in \rho^-(rc)} \lambda(x_1 x_2) \geq \sup_{x_2 \in \rho^-(c)} \lambda(x_2) = \rho(\lambda(c)).
\end{aligned}$$

Therefore, $\rho(\lambda)$ is a T -fuzzy subhypernear-module of M' .

Proposition 2.8 Let T be a t -norm and let $\rho : M \rightarrow M'$ be a homomorphism of hypernear-modules. If λ is a T -fuzzy subhypernear-module of M' , then $\rho^{-1}(\lambda)$ is a T -fuzzy subhypernear-module of M .

Proof Assume that λ is a T -fuzzy subhypernear-module of M' . Let $a, b \in M$. Then, we have

$$\begin{aligned}
\text{in} \rho \{ \rho^{-1}(\lambda(c)) \} &= \text{in} \rho \{ \lambda(\rho(c)) \} \geq \\
&T(\lambda(\rho(a)), \lambda(\rho(b))) = T(\rho^{-1}(\lambda)(a), \rho^{-1}(\lambda)(b)),
\end{aligned}$$

and

$$\begin{aligned}
\rho^{-1}(\lambda)(a) &= \lambda(\rho(a)) \leq \lambda(-\rho(a)) = \lambda(\rho(-a)) = \rho^{-1}(\lambda)(-a), \\
\rho^{-1}(\lambda)(a) &= \lambda(\rho(a)) \leq \lambda(r\rho(a)) = \lambda(\rho(ra)) = \rho^{-1}(\lambda)(ra),
\end{aligned}$$

for all $a \in M$ and $r \in R$. As a result, $\rho^{-1}(\lambda)$ is a T -fuzzy subhypernear-module of M .

Now, we will give direct product of T -fuzzy subhypernear-module.

Let T be a t -norm and λ_1 and λ_2 be fuzzy sets in M . Then the T -product of λ_1 and λ_2 , written $[\lambda_1 \cdot \lambda_2]_T$, is given by for all $a \in M$

$$[\lambda_1 \cdot \lambda_2]_T(a) = T(\lambda_1(a), \lambda_2(a)).$$

Let M_1 and M_2 be two hypernear-modules, then for all (a_1, b_1) and (a_2, b_2) in $M_1 \times M_2$ we define

$$\begin{aligned}
(a_1, b_1) + (a_2, b_2) &= \{(a, b) \mid a \in a_1 + a_2, b \in b_1 + b_2\}, \\
(r_1, r_2) \cdot (a_1, b_1) &= \{(r_1 a_1, r_2 b_1) \mid \}.
\end{aligned}$$

It is clear that $M_1 \times M_2$ is a subhypernear-module and we will give product of $M_1 \times M_2$.

Definition 2.2 Let T be a t -norm and λ and ν be fuzzy sets on hypernear-modules M_1 and M_2 , respectively. Then γ defined on $M_1 \times M_2$ by for all $(a, b) \in M_1 \times M_2$,

$$\gamma(a, b) = T(\lambda(a), \nu(b))$$

is a fuzzy set on $M_1 \times M_2$, which is defined by $\lambda \times \nu$.

Example 2.5 Consider hypernear-module M and fuzzy set λ in Example 2.2. Given $M_1 = \{1, 2, 3\}$ and $R = \{a, b, c\}$ be two non-empty sets with the following hyperoperation table.

+	1	2	3	and	.	1	2	3
1	{1}	{2}	{3}		a	{1}	{2}	{3}
2	{2}	{3}	{2, 3}		b	{1}	{1, 2}	{1, 3}
3	{3}	{2, 3}	{3}		c	{1}	{2, 3}	{1}

Then $(M_1, +, \cdot)$ is a hypernear-module. Define a fuzzy set $\nu : M_1 \rightarrow [0, 1]$ by

$$\nu(x) = \begin{cases} 0.7, & x = 1, \\ 0.5, & x = 2, \\ 0.6, & x = 3. \end{cases}$$

Then $\gamma = \lambda \times \nu$ is a fuzzy set over $M \times M_1$ and relation matrix of γ as follows

	1	2	3
e	0.5	0.5	0.5
a	0.4	0.4	0.4
b	0.54	0.5	0.54
c	0.6	0.5	0.6

Theorem 2.6 Let T be a t -norm and let $M = M_1 \times M_2$ be the direct product of hypernear-modules M_1 and M_2 . If λ and ν are T -fuzzy subhypernear-modules of M_1 and M_2 respectively, then $\gamma = \lambda \times \nu$ is a T -fuzzy subhypernear-module of M .

Proof Let $(a_1, b_1), (a_2, b_2) \in M_1 \times M_2$ such that $(a, b) = (a_1, b_1) + (a_2, b_2)$. Then, we get

$$\begin{aligned} \gamma(a, b) &= (\lambda \times \nu)(a, b) = T(\lambda(a), \nu(b)) = \\ &T(T(\lambda(a_1), \lambda(a_2)), T(\nu(b_1), \nu(b_2))) = \\ &T(T(T(\lambda(a_1), \lambda(a_2)), \nu(b_1), \nu(b_2))) = \\ &T(T(T(\lambda(a_1), \nu(b_1)), \lambda(a_2), \nu(b_2))) = \\ &T(T(\lambda(a_1), \nu(b_1)), T(\lambda(a_2), \nu(b_2))) = \\ &T((\lambda \times \nu)(a_1, b_1), (\lambda \times \nu)(a_2, b_2)) = \\ &T(\gamma(a_1, b_1), \gamma(a_2, b_2)). \end{aligned}$$

Thus,

$$\inf_{(a,b) \in (a_1,b_1) + (a_2,b_2)} \gamma(a, b) = \inf_{(a,b) \in (a_1,b_1) + (a_2,b_2)} (\lambda \times \nu)(a, b) \geqslant T((\lambda \times \nu)(a_1, b_1), (\lambda \times \nu)(a_2, b_2)) = T(\gamma(a_1, b_1), \gamma(a_2, b_2)).$$

Also, we get

$$\begin{aligned} \gamma(a, b) &= (\lambda \times \nu)(a, b) = T(\lambda(a), \nu(b)) \leqslant \\ &T(\lambda(-a), \nu(-b)) = (\lambda \times \nu)(-a, -b) = \gamma(-a, -b), \\ \gamma(a, b) &= (\lambda \times \nu)(a, b) = T(\lambda(a), \nu(b)) \leqslant \\ &T(\lambda(r_1 a), \nu(r_2 b)) = (\lambda \times \nu)(r_1 a, r_2 b) = \gamma(r_1 a, r_2 b). \end{aligned}$$

Consequently, $\gamma = \lambda \times \nu$ is a T -fuzzy subhypernear-module of M .

Theorem 2.7 Let T be a t -norm and let λ and ν be fuzzy sets of hypernear-modules M_1 and M_2 respectively. If $\lambda \times \nu$ is an idempotent T -fuzzy subhypernear-module of $M_1 \times M_2$, then at least one of the following two conditions are satisfied.

- (1) $\nu(0) \geq \lambda(a), \forall a \in M_1$;
- (2) $\lambda(0) \geq \nu(b), b \in M_2$.

Proof Assume that $\lambda \times \nu$ is an idempotent T -fuzzy subhypernear-module of $M_1 \times M_2$. Without loss of generality, we may admit that none of these conditions satisfied. Then $\exists a_0 \in M_1$ and $b_0 \in M_2$ such that

$$\lambda(a_0) > \nu(0) \text{ and } \nu(b_0) > \lambda(0).$$

It follows that

$$\begin{aligned} (\lambda \times \nu)(a_0, b_0) &= T(\lambda(a_0), \nu(b_0)) > T(\nu(0), \lambda(0)) = \\ &= T(\lambda(0), \nu(0)) = (\lambda \times \nu)(0, 0). \end{aligned}$$

This leads to a contradiction because by Example 2.4, we get

$$(\lambda \times \nu)(0, 0) \geq (\lambda \times \nu)(a_0, b_0).$$

Theorem 2.8 Let T be a t -norm and let λ , ν , and $\lambda \times \nu$ be fuzzy sets of hypernear-modules M_1 , M_2 , and $M_1 \times M_2$ respectively. If $\lambda \times \nu$ is an idempotent T -fuzzy subhypernear-module of $M_1 \times M_2$, then λ is an idempotent T -fuzzy subhypernear-module of M_1 or ν is an idempotent T -fuzzy subhypernear-module of M_2 .

Proof From Theorem 2.5, we admit that $\lambda(a) \leq \nu(0)$ for $\forall a \in M_1$, because $\lambda \times \nu$ is an idempotent T -fuzzy subhypernear-module of $M_1 \times M_2$. Then we infer that λ is a T -fuzzy subhypernear-module of M_1 . Let $a, b \in M_1$ with $\forall c \in a + b$, we get

$$\begin{aligned} \lambda(c) &= T(\lambda(c), 1) \geq T(\lambda(c), \nu(0)) = (\lambda \times \nu)(c, 0) \geq \\ &\geq \inf_{(c,0) \in (a,0)+(b,0)} (\lambda \times \nu)(c, 0) \geq \\ &= T((\lambda \times \nu)(a, 0), (\lambda \times \nu)(b, 0)) = \\ &= T(T(\lambda(a), \nu(0)), T(\lambda(b), \nu(0))) \geq \\ &= T(T(\lambda(a), \lambda(a)), T(\lambda(b), \lambda(a))) = \\ &= T(\lambda(a), T(\lambda(b), \lambda(a))) = \\ &= T(T(\lambda(a), \lambda(b), \lambda(a))) = T(\lambda(a), \lambda(b)). \end{aligned}$$

Hence $\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b))$. Also, we have for $a \in M_1$ and $r \in R$

$$\begin{aligned} \lambda(-a) &= T(\lambda(-a), 1) \geq T(\lambda(-a), \nu(0)) = \\ &= (\lambda \times \nu)(-a, 0) = (\lambda \times \nu)(-(a, 0)) \geq \\ &= (\lambda \times \nu)(a, 0) = T(\lambda(a), \nu(0)) \geq \\ &= T(\lambda(a), \lambda(a)) = \lambda(a), \end{aligned}$$

and we get

$$\begin{aligned} \lambda(ra) &= T(\lambda(ra), 1) \geq T(\lambda(ra), \nu(0)) = \\ &= (\lambda \times \nu)(ra, 0) = (\lambda \times \nu)(r(a, 0)) \geq \\ &= (\lambda \times \nu)(a, 0) = T(\lambda(a), \nu(0)) \geq \\ &= T(\lambda(a), \lambda(a)) = \lambda(a). \end{aligned}$$

Now, we characterize the product of T -fuzzy subhypernear-modules of hypernear-modules for $k \geq 2$.

Firstly, we characterize the domain of t -norm T to $\prod_{i=1}^k [0, 1]$ as follows.

Definition 2.3 The function $T_k : \prod_{i=1}^k [0, 1] \rightarrow [0, 1]$ is given by

$$T_k(a_1, a_2, \dots, a_k) = T(a_i, T_{k-1}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k)),$$

for all $1 \leq i \leq k$ such that $k \geq 2$, $T_2 = T$ and $T_1 = \text{id}(\text{identity})$.

By using induction method on k , we get the following lemmas.

Lemma 2.2 Let T be a t -norm and $\forall a_i, b_i \in [0, 1]$ such that $1 \leq i \leq k$, $k \geq 2$, we get

$$T_k(T(a_1, b_1), T(a_2, b_2), \dots, T(a_k, b_k)) = T(T_k(a_1, a_2, \dots, a_k), T_k(b_1, b_2, \dots, b_k)).$$

Lemma 2.3 Let T be a t -norm and $\forall a_1, a_2, \dots, a_k \in [0, 1]$ such that $k \geq 2$, we get

$$T_k(a_1, a_2, \dots, a_k) = T(\dots T(T(T(a_1, a_2), a_3, a_4), \dots, a_k) = T(a_1, T(a_2, T(a_3, \dots, T(a_{k-1}, a_k), \dots))).$$

Corollary 2.2 Let T be a t -norm and $M = \prod_{i=1}^k M_i$ be direct product of family of hypernear-modules $\{M_i \mid i = 1, 2, \dots, k\}$. If λ_i is a T -fuzzy subhypernear-modules of M_i with $1 \leq i \leq k$, then $\lambda = \prod_{i=1}^k \lambda_i$ given by for all $a = (a_1, a_2, \dots, a_k) \in M$,

$$\lambda(x) = \left(\prod_{i=1}^k \lambda_i \right) (a_1, a_2, \dots, a_k) = T_k(\lambda_1(a_1), \lambda_2(a_2), \dots, \lambda_k(a_k))$$

is a T -fuzzy subhypernear-modules of M .

Proof Let $a = (a_1, a_2, \dots, a_k)$, $b = (b_1, b_2, \dots, b_k) \in \prod_{i=1}^k M_i = M$. By Lemma 2.1, we get

$$\begin{aligned} \lambda(c) &= \left(\prod_{i=1}^k \lambda_i \right) (c) = \left(\prod_{i=1}^k \lambda_i \right) (c_1, c_2, \dots, c_k) = \\ &\left(\prod_{i=1}^k \lambda_i \right) (a_1, a_2, \dots, a_k) + (b_1, b_2, \dots, b_k) = \\ &\left(\prod_{i=1}^k \lambda_i \right) (a_1 + b_1, a_2 + b_2, \dots, a_k + b_k) = \\ &T_k(\lambda_1(a_1 + b_1), \lambda_1(a_2 + b_2), \dots, \lambda_k(a_k + b_k)) \geq \\ &T_k(T(\lambda_1(a_1), \lambda_1(b_1)), T(\lambda_2(a_2), \lambda_2(b_2)), \dots, T(\lambda_k(a_k), \lambda_k(b_k))) = \\ &T(T_k(\lambda_1(a_1), \lambda_2(a_2), \dots, \lambda_k(a_k)), T_k(\lambda_1(b_1), \lambda_2(b_2), \dots, \lambda_k(b_k))) = \\ &T\left(\left(\prod_{i=1}^k \lambda_i\right)(a_1, a_2, \dots, a_k), \left(\prod_{i=1}^k \lambda_i\right)(b_1, b_2, \dots, b_k)\right) = T(\lambda(a), \lambda(b)). \end{aligned}$$

Hence $\inf_{c \in a+b} \lambda(c) \geq T(\lambda(a), \lambda(b))$. Similarly, for all $a = (a_1, a_2, \dots, a_k) \in M$ and $r = (r_1, r_2, \dots, r_k) \in R$, we obtain

$$\begin{aligned} \lambda(-a) &= \left(\prod_{i=1}^k \lambda_i \right) (-a) = \left(\prod_{i=1}^k \lambda_i \right) (-a_1, -a_2, \dots, -a_k) = \\ &T_k(\lambda_1(-a_1), \lambda_2(-a_2), \dots, \lambda_k(-a_k)) \geq \\ &T_k(\lambda_1(a_1), \lambda_2(a_2), \dots, \lambda_k(a_k)) = \left(\prod_{i=1}^k \lambda_i \right) (a_1, a_2, \dots, a_k) = \lambda(a), \end{aligned}$$

and

$$\begin{aligned}\lambda(ra) &= \left(\prod_{i=1}^k \lambda_i \right) (ra) = \left(\prod_{i=1}^k \lambda_i \right) (r_1 a_1, r_2 a_2, \dots, r_k a_k) = \\ &T_k(\lambda_1(r_1 a_1), \lambda_2(r_2 a_2), \dots, \lambda_k(r_k a_k)) \geq \\ &T_k(\lambda_1(a_1), \lambda_2(a_2), \dots, \lambda_k(a_k)) = \left(\prod_{i=1}^k \lambda_i \right) (a_1, a_2, \dots, a_k) = \lambda(a).\end{aligned}$$

Consequently, $\lambda = \prod_{i=1}^k \lambda_i$ is a T -fuzzy subhypernear-modules of M .

3 Conclusion

The concept of T -fuzzy subhypernear-modules are studied by using a t -norm and some properties of them are discussed. Additionally, idempotent T -fuzzy subhypernear-module of hypernear-module and subhypernear-module of hypernear-module are compared and the relationship between them is given. Furthermore, image and inverge image of T -fuzzy subhypernear-modules under a mapping are studied. Also, $\lambda_1 \times \lambda_2$ is a T -fuzzy subhypernear-module of hypernear-modules is investigated. By using a t -norm, we construct T -fuzzy subhypernear-modules in the finite direct product of hypernear-modules. The acquired results might be applied other algebraic hyperstructures. Authors combine these coherent methods to achieve a variety of significant results about T -fuzzy subhypernear modules of hypernear modules. Depend on our work, we suggest some questions for researchers.

- (1) T -complex fuzzy subhypernear-modules of hypernear-modules can be studied.
- (2) (T, S) –intuitionistic fuzzy subhypernear-modules of hypernear-modules can be presented.
- (3) (T, S) –bipolar fuzzy subhypernear-modules of hypernear-modules can be identified.
- (4) Neutrosophic subhypernear-modules of hypernear modules with triangular norms can be clarified.
- (5) Vague subhypernear-modules of hypernear modules with triangular norms can be defined.

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