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Tracking Control of a Class of Fuzzy Dynamical Systems Under the Concept of Granular Differentiability

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ABSTRACT
In this work, the tracking control of a class of uncertain linear dynamical systems is investigated. The uncertainty is considered to be represented as fuzzy numbers, and hence, these uncertain dynamical systems are referred to as fuzzy linear dynamical systems, which are presented in the form of fuzzy differential equations (FDEs). The solution of an FDE is found using an approach called relative-distance-measure fuzzy interval arithmetic and under the granular differentiability concept. The control objective is to provide a control law such that the output of the system tracks a desired reference input in the presence of uncertainties. To this end, a theorem is proposed, which suggests that the control law should take the form of a feedback of fuzzy states with fuzzy gains and a fuzzy pre-compensator. However, since the fuzzy states of the system may not always be measurable, a fuzzy observer is designed for the estimation of such fuzzy states. It is also clearly shown that the generalized Hukuhara differentiability concept is unable for solving the problem examined in this study. Finally, the efficiency of the approach is examined for a plane landing control problem.

KEYWORDS
horizontal membership function; fuzzy control; fuzzy linear system; granular arithmetic; granular derivative

The tracking control has been one of the important problems in the realm of control theory. The aim in tracking control is to design a control structure that guarantees the output of closed-loop system tracks the desired reference input. So far, a large body of studies have been dealt with designing tracking control. However, most of them have assumed that the parameters, coefficients and/or the boundary conditions of dynamical systems are certain values. This is while, due to some reasons such as temperature changes, uncertainty in measuring and identification of parameters, the model of dynamical system may not be always determined precisely. Due to the mentioned reasons, in this study a design of tracking control for uncertain linear dynamical systems is studied.

Uncertain dynamical systems are referred to as those whose mathematical models usually come with differential equations in which some of the parameters are uncertain. Recently fuzzy differential equations (FDEs) as a branch of uncertain differential equations have attracted much attention. The uncertainty in FDEs is considered as fuzzy numbers that are a sort of knowledge of experts. A considerable number of studies have been conducted on fuzzy differential equations among which the most important is a review on fuzzy differential equations that recently Mazandarani et al.[1] elegantly has elaborated on all the aspects of

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fuzzy differential equations. A large number of possible applications of FDEs[1] among which we shall recall few of them as follows. Fuzzy delay differential equations under the concept of granular calculus has been studied[2]. The economic model of investing in a project under experts' different opinions using type-2 FDEs has been presented[3]. A non-recursive method for solving fuzzy partial differential equations has been proposed[4]. A collocation method[5] has been presented for finding the solution of fractional fuzzy differential equations. The fuzzy time optimal control problem under the concept of granular derivative has been solved[6] where the mathematical model of system includes fuzzy parameters. Analysing the Cerebrospinal fluid pressure through a mathematical model expressed by the use of type-2 fuzzy fractional differential equation has been investigated[7]. In addition, a system of FDEs as a mathematical model for diabetes mellitus has been presented with some studies on its stability[8].

FDEs have their own mathematical rules that are close to interval computations, and somewhat different from crisp (classic) mathematics rules. For example, in the context of fuzzy mathematics it is possible that $\tilde{u} - \tilde{u} \neq 0$ where $\tilde{u}$ is a fuzzy number. In the area of fuzzy mathematics there are mainly two approaches for solving FDEs: 1. Fuzzy standard interval arithmetic (FSIA), e.g., see Refs. [9, 10] where the notion of generalized Hukuhara differentiability has been applied; 2. Relative distance measure fuzzy interval arithmetic (RDM-FIA), e.g., see Refs. [11–13] where the concept of horizontal membership functions, introduced by Piegat et al.[14], has been utilized.

Applying fuzzy mathematics for designing fuzzy controller in order to control fuzzy dynamical systems has become an attracting topic in recent years. In this direction, designing an optimal control for a fuzzy linear dynamical system with fuzzy boundary conditions and parameters has been presented in Refs. [15, 16]. Specifically, Ref. [15] designing an optimal fuzzy controller has been investigated for a dynamical system whose boundary conditions are uncertain. Additionally, Ref. [16], fuzzy optimal control for a class of fuzzy dynamical with fuzzy coefficients and certain boundary conditions has been studied. Moreover, the existence of a solution to optimal control problem of a non-linear fuzzy dynamical system constrained by initial value has been presented in Ref. [17]. Additionally, in Ref. [18], the Pontryagin maximum principle has been used to obtain a necessary optimality condition based on which the fuzzy optimal control function for a class of linear fuzzy dynamical systems is obtained. However, in the mentioned papers and some other papers, concepts of fuzzy derivatives have been used which are based on FSIA approach and then they suffer from many shortcomings. In the sequel we explain briefly a few of the drawbacks that stem from these concepts, for more details see Refs. [1, 19].

Designing a fuzzy controller for fuzzy dynamical systems has to do with fuzzy differential equations and one of the fuzzy mathematical approaches. Since dealing with FDEs needs a concept of fuzzy differentiability, such a concept is categorized based on the approach by which it is defined. The notions of strongly generalized Hukuhara derivative (SGH-derivative), generalized Hukuhara derivative (gH-derivative), and generalized derivative (g-derivative) are concepts of fuzzy differentiability which have been defined based on FSIA approach. Although the FSIA-based fuzzy derivatives have been applied in a large body of literature, recently for the first time, Mazandarani et al.[1, 19] have shed light on a considerable number of drawbacks that come with these notions. Specifically, in Ref. [1], it has been explained that SGH-derivative, gH-derivative, g-derivative and all the concepts that are equivalent or based on the mentioned derivatives (integer order or fractional order) have at least 6 drawbacks among which is disability in factorization and unnatural behaviour in modelling (UBM) phenomenon. Disability in factorization as it has been explained in Ref. [1] means that based on FSIA-based approaches, i.e. SGH-derivative, gH-derivative, g-derivative, a simple FDE such as $\dot{\tilde{x}}(t) = \tilde{a}\tilde{x}(t) + \tilde{a}\tilde{f}(t)$ may have a solution that is different from $\dot{\tilde{x}}(t) = \tilde{a}\left(\tilde{x}(t) + \tilde{f}(t)\right)$. The UBM phenomenon is referred to that one in which different guises of a same FDE may have different solutions. For example, based on SGH-derivative, or gH-derivative, the following FDEs may have different solutions $\dot{\tilde{x}}(t) = \tilde{a}\tilde{x}(t) + \tilde{a}\tilde{f}(t)$, $\dot{\tilde{x}}(t) - \tilde{a}\tilde{x}(t) = \tilde{a}\tilde{f}(t)$, $\dot{\tilde{x}}(t) - \tilde{a}\tilde{f}(t) = \tilde{a}\tilde{x}(t)$. For more details about other drawbacks of SGH-derivative, gH-derivative, g-derivative one may refer to Refs. [1, 19]. To overcome the mentioned drawbacks, Mazandarani et al.[19] introduced the concept of granular differentiability (gr-derivative) and granular calculus. The gr-derivative recently has become and continues to be the most important and powerful concept by which the applicability of FDEs and their analysis can be done more effectively than ever before. What follows briefly review some of the recent applications of gr-derivative.

1 Applications of Granular Derivative

In the direction of using gr-derivative, a sub-optimal control for fuzzy linear dynamical systems has been proposed in Ref. [20]. The fuzzy time optimal control on the basis of gr-derivative and RDM-FIA has been studied in Ref. [6]. Granular fuzzy PID control as a combination of the concept of PID controller and granular derivative and granular integral has been proposed[21]. In Ref. [22], the existence and uniqueness results have been obtained for impulsive fuzzy fractional stochastic differential system (FSDS) with granular derivative and using contraction principle. In Ref. [23], a class of granular type optimality guidelines for the fuzzy multi-objective optimizations based on vector granular convexity and granular differentiability has been introduced. The study of the fractional fuzzy option pricing model by solving the fuzzy time-fractional Black–Scholes European option pricing equations under the concept of granular derivative has been presented[24]. In Refs. [25, 26], under the concept of granular differentiation, the fuzzy optimization problems with the general fuzzy function as the objective function for type-1 and type-2 fuzzy functions have been investigated. On the basis of the powerful notion of granular derivative, the consensus problem for linear multi-agent systems with granular fuzzy uncertainty has been investigated[27, 28]. In Ref. [29], the stability and controllability of fuzzy singular dynamical systems under the concept of granular calculus has been studied. The granular difference and granular division operations properties have been presented[30, 31]. The existence and uniqueness of solutions for homogeneous and non-homogeneous system of first order linear fuzzy boundary value problems under granular differentiability has been discussed[32]. Moreover, the solution of second-order fuzzy homogeneous differential equations with fuzzy coefficients have been studied[33]. As the other research works in the context of control theory in which the concept of gr-derivative has applied we shall mention fuzzy production inventory control model[34, 35], optimal control of susceptible exposed infections recovered epidemic[36], stability of fuzzy linear dynamical systems[37, 38], regulation problem[39], fuzzy optimal control for COVID-19 epidemic[40], fuzzy variational problem and optimal control[41]. This article aims to design a tracking controller for a fuzzy linear dynamical system, assuming that the fuzzy states of the system need to be estimated. To estimate the fuzzy states, a fuzzy observer is designed. The control law consists of a fuzzy feedback of states and a static pre-compensator. A theorem with its proof is provided, assuring us that the output of the system satisfactorily tracks the reference input in the presence of uncertainty. In other words, the proposed method is based on fuzzy mathematics, which allows us to consider uncertainty in the system's mathematical model parameters as fuzzy sets. This viewpoint enables us to obtain state and control trajectories with some information about the possibility distribution of the states and control. Additionally, the proposed method is based on the concept of granular calculus, which overcomes the drawbacks that exist in strongly generalized Hukuhara, generalized Hukuhara, generalized differentiability concepts, or all the methods that are based on fuzzy standard interval arithmetic. Some of these drawbacks include multiplicity of solutions, doubling property, unnatural behavior in modelling (UBM) phenomenon, and factorization disability. More details about the drawbacks can be found in Refs. [1, 19].

In the rest of this research work some basic concepts related to fuzzy mathematics bases are given in Section 2, the design procedure of controller and observer is presented in Section 3. Section 4 is devoted to the example and results, Section 5 is simulation. Section 6 is discussion, and Section 7 is conclusion.

2 Mathematical Modelling of Fuzzy Functions

Throughout this paper, the set of all real numbers is denoted by $\mathbb{R}$, and the set of all the type-1 fuzzy numbers on $\mathbb{R}$ by E_1. The left and right end-points of $\alpha-$level sets of the fuzzy set $\tilde{A}$, are denoted by $\bar{A}^\alpha$ and A^α, respectively. The transpose of a matrix $\boldsymbol{Y}=\left[y_{ij}\right]_{n\times n}$, $i, j=1, 2, \ldots, n$ is denoted by $\boldsymbol{Y}^{\mathrm{T}}$.

Definition 2.1[11,19] Let $\tilde{u}:[a, b]\subseteq\mathbb{R}\rightarrow[0,1]$ be a fuzzy number. The horizontal membership function $u^{\mathrm{gr}}=[0,1]\times[0,1]\rightarrow[a,b]$ is a representation of $\tilde{u}(x)$ as $u^{\mathrm{gr}}(\alpha,\beta_u)=x$ in which "gr" stands for the granule of information included in $x\in[a, b]$, $\beta\in[0,1]$ is the membership degree of x in $\tilde{u}(x)$, $\beta_u\in[0,1]$ is called relative-distance-measure (RDM) variable, and $u^{\mathrm{gr}}(\alpha,\beta_u)=\underline{u}^\alpha+(\bar{u}^\alpha-\underline{u}^\alpha)\beta_u$.

Note 1[19] For the sake of simplicity, the horizontal membership function of $\tilde{u}(x)\in E_1$ is also denoted by:

$$H(\tilde{u})\stackrel{\Delta}{=}u^{\mathrm{gr}}(\alpha,\beta_u) \tag{1}$$

Note 2[19] The $\alpha-$level sets of $\tilde{u}(x)\in E_1$ which are in fact the span of the information granule can be obtained using

$$\mathscr{H}^{-1}(u^{\mathrm{gr}}(\alpha,\beta_u))=[\tilde{u}]^\alpha=\left[\inf_{\lambda\geqslant\alpha}\ \min_{\beta_u}\ u^{\mathrm{gr}}(\lambda,\beta_u)\ \ \sup_{\lambda\geqslant\alpha}\ \max_{\beta_u}\ u^{\mathrm{gr}}(\lambda,\beta_u)\right] \tag{2}$$

Definition 2.2[19] Two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ are said to be equal if and only if $\mathscr{H}(\tilde{u})=\mathscr{H}(\tilde{v})$ for all $\beta_u=\beta_v\in[0,1]$.

Definition 2.3[19] Let $\tilde{f}:[a,b]\subseteq\mathbb{R}\rightarrow E_1$ include n distinct fuzzy numbers $\tilde{u}_1,\tilde{u}_2,\ldots,\tilde{u}_n$. The horizontal membership function of $\tilde{f}(t)$ is denote by $\mathscr{H}(\tilde{f}) = f^{\mathrm{gr}}\left(t, \alpha, \beta_f\right)$, and defined as

$$f^{\mathrm{gr}} : [a, b] \times [0, 1] \times [0, 1] \times \underbrace{\ldots}_{n} \times [0, 1] \rightarrow [c, d] \subseteq \mathbb{R} \tag{3}$$

Definition 2.4[19] The fuzzy function $\tilde{f} : [a, b] \subseteq \mathbb{R} \rightarrow E_1$ is said to be granular differentiable (gr-differentiable) at $t \in [a, b]$ if there exists a fuzzy number ${}^{\mathrm{gr}}D\tilde{f}(t) \in E_1$ such that the following limit exists:

$$\lim_{h \to 0} \frac{\tilde{f}(t+h) - \tilde{f}(t)}{h} = {}^{\mathrm{gr}}D\tilde{f}(t) \tag{4}$$

Theorem 2.1[19] The fuzzy function $\tilde{f} : [a, b] \subseteq \mathbb{R} \rightarrow E_1$ is granular differentiable at the point $t \in [a, b]$ if and only if its horizontal membership function is differentiable with respect to t at that point. Moreover,

$$\mathscr{H}\left({}^{\mathrm{gr}}D\tilde{f}(t)\right) = \frac{\partial f^{\mathrm{gr}}\left(t,\ \alpha,\ \beta_f\right)}{\partial t}.$$

Definition 2.5[19] Let $\tilde{f} : [a, b] \subseteq \mathbb{R} \rightarrow E_1$ mapping $t \rightarrow \tilde{f}(t)$ be a continuous fuzzy function whose horizontal membership function, i.e., $f^{\mathrm{gr}}(t, \alpha, \beta_u)$, is integrable on $t \in [a, b]$. Let $\int_a^b \tilde{f}(t)\,\mathrm{d}t$ denote the granular integral of $\tilde{f}(t)$ on $[a, b]$. Then, the fuzzy function $\tilde{f}(t)$ is said to be granular integrable (gr-integrable) on $[a, b]$ if there exists a fuzzy number $\tilde{m} = \int_a^b \tilde{f}(t)\,\mathrm{d}t$ such that:

$$\mathscr{H}(\tilde{m}) = \int_a^b \mathscr{H}\left(\tilde{f}(t)\right)\mathrm{d}t \tag{5}$$

3 Design of Fuzzy State Feedback and Observer for Fuzzy Linear Dynamical Systems

3.1 Design of fuzzy state feedback for fuzzy linear dynamical systems

Consider the following fuzzy linear dynamical system:

$$\begin{cases} {}^{\mathrm{gr}}D\tilde{X}(t) = \tilde{\boldsymbol{A}}\tilde{X}(t) + \tilde{\boldsymbol{B}}\tilde{u}(t) \\ \tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{C}}\tilde{X}(t) \\ \tilde{X}(t_0) = \tilde{X}_0 \end{cases} \tag{6}$$

where $\tilde{X}(t) \triangleq [\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)]^{\mathrm{T}}, \tilde{X} : [t_0, t_f] \subseteq R^+ \cup \{0, \infty\} \rightarrow E_1$ is the states vector of the system, $\tilde{\boldsymbol{u}} : [t_0, t_f] \subseteq \mathbb{R}^+ \cup \{0, \infty\} \rightarrow E_1$ is the control input. The output of the system has been denoted by $\tilde{\boldsymbol{y}} : [t_0, t_f] \subseteq \mathbb{R}^+ \cup \{0, \infty\} \rightarrow E_1$, and ${}^{\mathrm{gr}}D\tilde{X}(t) \triangleq [{}^{\mathrm{gr}}Dx_1(t), {}^{\mathrm{gr}}Dx_2(t), \ldots, {}^{\mathrm{gr}}Dx_n(t)]^{\mathrm{T}}$ in which the derivatives are in the sense of gr-derivative. The matrices $\tilde{\boldsymbol{A}} = [\tilde{a}_{ij}]_{n \times n}, i, j = 1, 2, \ldots, n; \tilde{\boldsymbol{B}} = [\tilde{b}_{ik}]_{n \times 1}$, $i = 1, 2, \ldots, n; k = 1$ are fuzzy matrices. The fuzzy tracking control problem aims at finding a fuzzy controller $\tilde{u}$ based on which the output of the fuzzy system tracks a pre-defined reference input.

Theorem 3.1[42] Consider fuzzy linear dynamical system (Eq. (6)). Suppose there is a fuzzy matrix $\tilde{K} = [\tilde{\boldsymbol{k}}_1, \tilde{\boldsymbol{k}}_2, \ldots, \tilde{\boldsymbol{k}}_n]$ such that fuzzy eigenvalues of the fuzzy matrix $\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{K}$ are negative fuzzy numbers. The output of the system tracks the reference input r, i.e., $\lim_{t \to \infty} \tilde{\boldsymbol{y}}(t) = r$, if the fuzzy control input is in the form of

$$\begin{aligned}\tilde{\boldsymbol{u}}(t) &= -\tilde{K}\tilde{X}(t) - \left[C\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{K}\right)^{-1}\tilde{\boldsymbol{B}}\right]^{-1} r, \\ G_{cl}(0)^{-1} &= \left[\tilde{\boldsymbol{C}}\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{K}\right)^{-1}\tilde{\boldsymbol{B}}\right]^{-1}\end{aligned} \tag{7}$$

Proof It is given in Ref. [42].

The block diagram of this control system is shown in Fig. 1.

3.2 Design of fuzzy observer for fuzzy linear dynamical systems

In the previous section, what presented the design of control systems with mode feedback. For the design of a state feedback controller, all mode variables must be available for feedback. In many real applications, it is not possible to measure all state variables. In these cases, estimation of state variables is required to implement state feedback. For this purpose, a fuzzy observer needs to be designed to estimate the fuzzy system states. It should be noted that the input variables are always fuzzy values. Assume that the fuzzy dynamic system is given as Eq. (8).

$$\begin{aligned}{}^{\mathrm{gr}}D\tilde{X}(t) &= \tilde{\boldsymbol{A}}\tilde{X}(t) + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}}(t) \\ \tilde{\boldsymbol{y}}(t) &= \tilde{\boldsymbol{C}}\tilde{X}(t)\end{aligned} \tag{8}$$

The state feedback control system $\tilde{\boldsymbol{u}}(t) = -\tilde{K}\tilde{X}(t)$ in the previous section was designed with the assumption of availability $\tilde{X}(t)$. However, it is assumed that instead of being able to measure system states, the system output can only be measured as follows:

$$\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{C}}\tilde{X}(t) \tag{9}$$

Consider the fuzzy dynamic system as shown in Fig. 2, whose inputs $\tilde{u}(t), \tilde{y}(t)$ and outputs are $\tilde{x}(t)$.

The action of state space of this dynamic system, which is the system observer, is

$${}^{\mathrm{gr}}D\hat{\tilde{X}}(t) = \hat{\tilde{A}}\hat{\tilde{X}}(t) + \hat{\tilde{B}}\tilde{\boldsymbol{u}}(t) + \tilde{\boldsymbol{L}}\tilde{\boldsymbol{y}}(t) \tag{10}$$

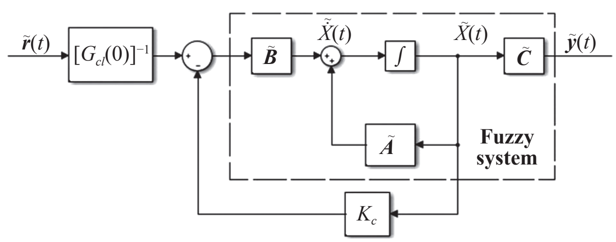


Fig. 1 Fuzzy state feedback structure.

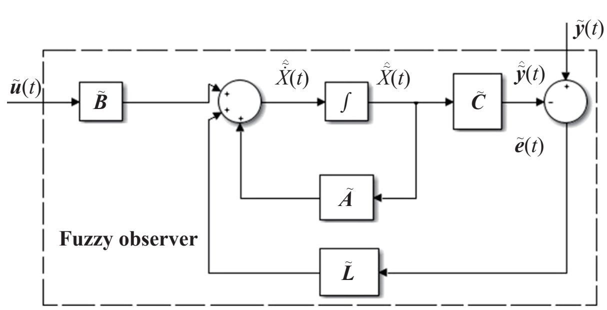


Fig. 2 Overview of fuzzy linear observer.

where $\hat{\hat{X}}$ is n-dimensional vector, and the matrices $\hat{\hat{A}}$, $\hat{\hat{B}}$, and $\tilde{L}$ must be selected so that the following observer error is asymptotically zero.

$$\tilde{\boldsymbol{e}}(t) = \tilde{X}(t) - \hat{\hat{X}}(t) \tag{11}$$

By the use of Eqs. (8)–(11), a dynamic equation for error can be obtained as follows:

$$\begin{aligned} &{}^{gr}D\tilde{\boldsymbol{e}}(t) = {}^{gr}D\tilde{X}(t) - {}^{gr}D\hat{\hat{X}}(t) = \\ &\tilde{\boldsymbol{A}}\tilde{X}(t) + \tilde{\boldsymbol{B}}\tilde{u}(t) - \hat{\hat{A}}\left[\tilde{X}(t) - \tilde{\boldsymbol{e}}(t)\right] - \hat{\hat{B}}\tilde{\boldsymbol{u}}(t) - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\tilde{X}(t) = \\ &\hat{\hat{A}}\,\tilde{\boldsymbol{e}}(t) + \left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}} - \hat{\hat{A}}\right)\tilde{X}(t) + \left(\tilde{\boldsymbol{B}} - \hat{\hat{B}}\right)\tilde{\boldsymbol{u}}(t) \end{aligned} \tag{12}$$

If we want the error of the observer to be independent of $\tilde{X}(t)$ and $\tilde{\boldsymbol{u}}(t)$ asymptotically zero, we must set the coefficients $\tilde{X}(t)$ and $\tilde{\boldsymbol{u}}(t)$ in Eq. (12) equal to zero and correspond to a stable dynamic system; Therefore

$$\hat{\hat{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}} \tag{13}$$

and

$$\hat{\hat{B}} = \tilde{\boldsymbol{B}} \tag{14}$$

Hence, the choice of $\hat{\hat{A}}$, $\hat{\hat{B}}$, and $\tilde{\boldsymbol{L}}$ is not desirable. There is no choice in the selection, and it must be equal to the control matrix $\tilde{\boldsymbol{B}}$. $\tilde{\boldsymbol{A}}$ is also determined after selection. $\tilde{\boldsymbol{L}}$, the only matrix that can be selected is the matrix $\tilde{\boldsymbol{L}}$.

By entering Eqs. (13) and (14) in Eq. (10), we have

$${}^{gr}D\hat{\hat{X}}(t) = \left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\right)\hat{\hat{X}}(t) + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}}(t) + \tilde{\boldsymbol{L}}\tilde{\boldsymbol{y}}(t) = \left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\right)\hat{\hat{X}}(t) + \left[\tilde{\boldsymbol{B}}\tilde{\boldsymbol{L}}\right]\begin{bmatrix}\tilde{\boldsymbol{u}}(t)\\ \tilde{\boldsymbol{y}}(t)\end{bmatrix} \tag{15}$$

The equation that describes the observer (Eq. (15)) is the equation of the system's state (Eq. (8)). This equation can be written as follows:

$${}^{gr}D\hat{\hat{X}}(t) = \tilde{\boldsymbol{A}}\hat{\hat{X}}(t) + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}}(t) + \tilde{\boldsymbol{L}}\left[\tilde{\boldsymbol{y}}(t) - \tilde{\boldsymbol{C}}\hat{\hat{X}}(t)\right] \tag{16}$$

The matrix $\hat{\hat{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}$ is the system's state, which also appears in Eq. (12) of the dynamic error equation. By satisfying Eqs. (13) and (14), Eq. (12) will be as follows:

$${}^{gr}D\tilde{\boldsymbol{e}}(t) = \hat{\hat{A}}\tilde{\boldsymbol{e}}(t) \tag{17}$$

For the error to be asymptotically close to zero, $\hat{\hat{A}}$ must be a stable matrix; that is, the eigenvalues of the closed-loop matrix $\hat{\hat{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\,\tilde{\boldsymbol{C}}$ are to the left of the imaginary axis.

Definition 3.1 Suppose $\tilde{\boldsymbol{y}}(t, t_0, \tilde{X}_0, \tilde{u})$ the output variable responds to a fuzzy linear dynamic system (Eq. (18)) for the input $\tilde{\boldsymbol{u}}(t)$ and initial state $\tilde{X}(t_0) = \tilde{X}_0$. The system given by Eq. (18) is then called finite-grained visibility if it can be computed uniquely $\tilde{X}(t_0)$ by the availability of information $\tilde{\boldsymbol{u}}(t)$ and $\tilde{\boldsymbol{y}}(t)$ over a period of time.

$$\begin{aligned} &{}^{gr}D\tilde{X}(t_0) = \tilde{\boldsymbol{A}}\tilde{X}(t_0) + \tilde{\boldsymbol{B}}\tilde{u}(t), \\ &\tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{C}}\tilde{X}(t_0) \end{aligned} \tag{18}$$

Proposition 3.1 Consider the fuzzy dynamic system (Eq. (18)) and assume that $A^{gr}(\alpha,\beta_A)$ and $C^{gr}(\alpha,\beta_C)$ the horizontal membership functions are matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{C}}$. In this case, the system (Eq. (18)) or pair $(\tilde{\boldsymbol{A}},\tilde{\boldsymbol{C}})$ is visible fine-grained if and only if it has a fine-grained matrix (Eq. (19)) per $\alpha,\beta_A,\beta_C\in[0,1]$ complete rank.

$$\left[C^{gr}(\alpha,\beta_C),\ C^{gr}(\alpha,\beta_C)A^{gr}(\alpha,\beta_A),\ C^{gr}(\alpha,\beta_C)\left(A^{gr}(\alpha,\beta_A)\right)^2,\ldots,\ C^{gr}(\alpha,\beta_C)\left(A^{gr}(\alpha,\beta_A)\right)^{n-1}\right]^{\mathrm{T}} \tag{19}$$

3.3 Fuzzy tracker control for linear fuzzy dynamic systems with the integral based observer

An overview of the tracking system with mode feedback and integral mode observer is shown in Fig. 3 where the integral operators are granular integrals defined in Definition 5.

The closed-loop system will be $(2n+1)$ dimensions. The equations of the compensation system are

$$\begin{aligned}
&{}^{gr}D\tilde{\boldsymbol{z}}(t)=\tilde{\boldsymbol{r}}(t)-\tilde{\boldsymbol{y}}(t),\\
&{}^{gr}D\hat{\tilde{\boldsymbol{x}}}(t)=(\tilde{\boldsymbol{A}}-\tilde{\boldsymbol{l}}\tilde{\boldsymbol{C}})\hat{\tilde{\boldsymbol{x}}}(t)+\tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}}(t)+\tilde{\boldsymbol{l}}\tilde{\boldsymbol{y}}(t),\\
&\tilde{\boldsymbol{u}}(t)=-\tilde{\boldsymbol{k}}\hat{\tilde{\boldsymbol{x}}}(t)+\tilde{K}_{n+1}\tilde{\boldsymbol{z}}(t)
\end{aligned} \tag{20}$$

The equations of the closed-loop system are obtained by combining Eqs. (8) and (20) as follows:

$$\begin{aligned}
&\begin{bmatrix}{}^{gr}D\tilde{\boldsymbol{x}}(t)\\ {}^{gr}D\tilde{\boldsymbol{z}}(t)\\ {}^{gr}D\hat{\tilde{\boldsymbol{x}}}(t)\end{bmatrix}=\begin{bmatrix}\tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}}\tilde{K}_{n+1} & -\tilde{\boldsymbol{B}}\tilde{k}\\ -\tilde{\boldsymbol{C}} & 0 & 0\\ \tilde{\boldsymbol{l}}\tilde{C} & \tilde{\boldsymbol{B}}\tilde{K}_{n+1} & \tilde{\boldsymbol{A}}-\tilde{\boldsymbol{B}}\tilde{k}-\tilde{\boldsymbol{l}}\tilde{C}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\ \tilde{\boldsymbol{z}}(t)\\ \hat{\tilde{\boldsymbol{x}}}(t)\end{bmatrix}+\begin{bmatrix}0\\ \boldsymbol{I}\\ 0\end{bmatrix}\tilde{\boldsymbol{r}}(t),\\
&\tilde{\boldsymbol{y}}(t)=\begin{bmatrix}\tilde{\boldsymbol{C}} & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{x}}(t)\\ \tilde{\boldsymbol{z}}(t)\\ \hat{\tilde{\boldsymbol{x}}}(t)\end{bmatrix}
\end{aligned} \tag{21}$$

Now by applying the following conversion:

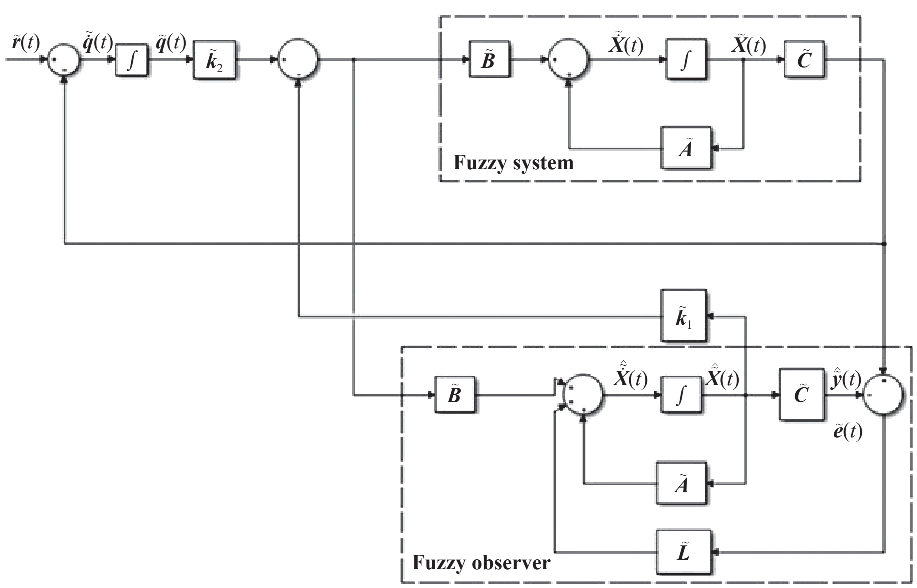


Fig. 3 State feedback structure with integral control and observer for fuzzy system.

$$\begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{z}}(t) \\ \tilde{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \boldsymbol{I} & 0 \\ \boldsymbol{I} & 0 & -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{z}}(t) \\ \hat{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} \tag{22}$$

The closed-loop equations of the system are

$$\begin{gathered} \begin{bmatrix} {}^{\mathrm{gr}}D\tilde{\boldsymbol{x}}(t) \\ {}^{\mathrm{gr}}D\tilde{\boldsymbol{z}}(t) \\ {}^{\mathrm{gr}}D\tilde{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{k}} & \tilde{\boldsymbol{B}}\tilde{K}_{n+1} & \tilde{\boldsymbol{B}}\tilde{\boldsymbol{k}} \\ -\tilde{\boldsymbol{C}} & 0 & 0 \\ 0 & 0 & \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{z}}(t) \\ \tilde{\boldsymbol{e}}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{I} \\ 0 \end{bmatrix} \tilde{\boldsymbol{r}}(t), \\ \tilde{\boldsymbol{y}}(t) = \begin{bmatrix} \tilde{\boldsymbol{C}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{z}}(t) \\ \tilde{\boldsymbol{e}}(t) \end{bmatrix} \end{gathered} \tag{23}$$

The closed-loop system have $(2n+1)$ the desired stable pole with the right choice of the vector $\begin{bmatrix} \tilde{\boldsymbol{k}} - \tilde{K}_{n+1} \end{bmatrix}$. The reason comes from the assumption on the controllability of the system as proved in the article[42] and the observability of the system given by Eq. (8). In addition, it has been assumed that the system is integral controllable. Therefore, we have a steady-state step $\tilde{\boldsymbol{r}}(t) = \tilde{\boldsymbol{r}}$ for reference input.

$$\tilde{\boldsymbol{u}}_{ss} = -\tilde{\boldsymbol{k}}(\tilde{\boldsymbol{x}}_{ss} - \tilde{\boldsymbol{e}}_{ss}) + \tilde{K}_{n+1}\tilde{\boldsymbol{z}}_{ss} \tag{24}$$

where *ss* subtitle indicates steady state. Equation (23) in the steady-state is

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{k}} & \tilde{\boldsymbol{B}}\tilde{K}_{n+1} & \tilde{\boldsymbol{B}}\tilde{\boldsymbol{k}} \\ -\tilde{\boldsymbol{C}} & 0 & 0 \\ 0 & 0 & \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{l}}\tilde{\boldsymbol{C}} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{z}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{I} \\ 0 \end{bmatrix} \tilde{\boldsymbol{r}} \tag{25}$$

we now define the following similarity conversion using Eq. (24):

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ -\tilde{\boldsymbol{k}} & \tilde{K}_{n+1} & \tilde{\boldsymbol{k}} \\ 0 & 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{z}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{u}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} \tag{26}$$

or

$$\begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ \tilde{K}_{n+1}^{-1}\tilde{\boldsymbol{k}} & \tilde{K}_{n+1}^{-1} & -\tilde{K}_{n+1}^{-1}\tilde{\boldsymbol{k}} \\ 0 & 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{u}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{z}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} \tag{27}$$

which we have by applying it in Eq. (25),

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}} & 0 \\ -\tilde{\boldsymbol{C}} & 0 & 0 \\ 0 & 0 & \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{l}}\tilde{\boldsymbol{C}} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{ss} \\ \tilde{\boldsymbol{u}}_{ss} \\ \tilde{\boldsymbol{e}}_{ss} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{I} \\ 0 \end{bmatrix} \tilde{\boldsymbol{r}} \tag{28}$$

which gives the following two equations:

$$\begin{bmatrix} \tilde{\boldsymbol{A}} & \tilde{\boldsymbol{B}} \\ -\tilde{\boldsymbol{C}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{X}}_{ss} \\ \tilde{\boldsymbol{u}}_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{I} \end{bmatrix} \tilde{\boldsymbol{r}} \tag{29}$$

and

$$\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{l}}\tilde{C}\right)\tilde{\boldsymbol{e}}_{ss} = 0 \tag{30}$$

Given the integral controllability, the system gives Eq. (29).

$$\begin{bmatrix} \tilde{X}_{ss} \\ \tilde{u}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \tilde{r} \end{bmatrix} \tag{31}$$

And assuming that the system does not have a polar open loop at the origin,

$$\begin{bmatrix} \tilde{x}_{ss} \\ \tilde{u}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{A}^{-1}\tilde{B}\left(\tilde{C}\tilde{A}^{-1}\tilde{B}\right)^{-1}\tilde{r} \\ -\left(\tilde{C}\tilde{A}^{-1}\tilde{B}\right)^{-1}\tilde{r} \end{bmatrix} \tag{32}$$

On the other hand, Eq. (30) shows that $\tilde{e}_{ss} = 0$ because $\tilde{A} - l\tilde{\tilde{C}}$ has particular values in the left half of the page. Therefore, we have Eqs. (32) and (24).

$$\tilde{z}_{ss} = \tilde{K}_{n+1}^{-1}\left(\tilde{u}_{ss} + \tilde{k}\tilde{x}_{ss}\right) = \tilde{K}_{n+1}^{-1}\left[I - \tilde{k}\tilde{A}^{-1}\tilde{B}\right]G(0)^{-1}\tilde{r} \tag{33}$$

where $G(s)$ is the open-loop conversion function of the system. We also have

$$\tilde{y}_{ss} = \tilde{C}\tilde{x}_{ss} = \tilde{C}\left[\tilde{A}^{-1}\tilde{B}\left(\tilde{C}\tilde{A}^{-1}\tilde{B}\right)^{-1}\tilde{r}\right] = \tilde{r} \tag{34}$$

Therefore, tracking is done in a steady state.

4 Comparison of Other Approaches with the Proposed Method

What follows draws a comparison of the other approaches with the one that has been already introduced. Indeed, this section provides some explanations about the fact that why the problem of tracking control of fuzzy dynamical systems cannot be solved by the use of FSIA-based methods. As we have discussed in the introduction section, the methods dealing with fuzzy dynamical systems can be categorized as those which are based on the so-called FSIA-based approaches and RDM-FIA-based method. As the former is concerned, one may apply H-derivative, SGH-derivative, gH-derivative, or g-derivative with their corresponding calculus and the fuzzy standard interval arithmetic operations. As an instance, let us assume that someone has considered one of the concepts related to the FSIA-based approaches, e.g. SGH-derivative, and the aim is to design the fuzzy observer as it has been done in subsection 3.2. Then, in order to design an observer, the method that has been applied must assure us that the observer error shown in Eq. (11), i.e.,

$$\tilde{e}(t) = \tilde{X}(t) - \hat{\tilde{X}}(t) \tag{35}$$

tends to zero as $\tilde{X}(t)$ approaches $\hat{\tilde{X}}(t)$, and it is equal to zero when $\tilde{X}(t)$ is equal to $\hat{\tilde{X}}(t)$. However, unlike the RDM-FIA-based method, FSIA-based approaches are not capable of providing such a guarantee. More specifically, according to the FSIA-based approaches the Eq. (35) is not equal to zero even if $\tilde{X}(t)$ is equal to $\hat{\tilde{X}}(t)$. As an example, let us suppose that $\tilde{X}(t)$ and $\hat{\tilde{X}}(t)$ are both equal to the triangular fuzzy number $(20, 30, 40)$ at $t = 2$. Then, the observer error, based on the FSIA-based approaches, is obtained as $\tilde{e}(t) = (-20, 0, 20)$ at $t = 2$ which is not equal to zero. Nevertheless, by the use of RDM-FIA-based method, we have $H(\tilde{e}(t)) = 20 + 10\alpha + 20\beta(1-\alpha) - [20 + 10\alpha + 20\beta(1-\alpha)] = 0$ reading that $\tilde{e}(t) = 0$

at $t = 2$. Accordingly, we cannot obtain the relationship (13), i.e.,

$$\hat{\tilde{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}$$

from

$$\hat{\tilde{A}} - \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}} = 0 \tag{36}$$

Due to the fact that the operations in the left hand side of (36) is not equal to zero based on the FSIA-based approaches. This is also the case for many other relationships, e.g., the Eq. (30) in subsection 3.3. Therefore, using the FSIA-based approaches the tracking control problem of fuzzy dynamical systems cannot be dealt with. In addition, there are some more drawbacks among which is the disability in the factorization that has been discussed in Refs. [1, 43]. In order to clarify why such a shortcoming makes FSIA-based approaches unable as a tool for solving the tracking control problem of fuzzy dynamical systems let us concentrate on the relationship (12) where the factorized term $-\hat{\tilde{A}}\left[\tilde{X}(t) - \tilde{\boldsymbol{e}}(t)\right]$ has been expanded to $-\hat{\tilde{A}}\tilde{X}(t) + \hat{\tilde{A}}\tilde{\boldsymbol{e}}(t)$ and the term $\tilde{\boldsymbol{A}}\tilde{X}(t) - \hat{\tilde{A}}\tilde{X}(t) - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}}\tilde{X}(t)$ has been factorized as $\left(\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{L}}\tilde{\boldsymbol{C}} - \hat{\tilde{A}}\right)\tilde{X}(t)$ by the use of RDM-FIA based method. However, such operations cannot be done by the use of FSIA-based approaches because they do not hold the property of the factorization in a general setting. As the simplest example, let us assume that $\tilde{x}$ is a triangular fuzzy number equal to $(1, 2, 3)$. Then, based on FSIA-based approaches $4\tilde{x} - 3\tilde{x} = (-5, 2, 9)$. Nevertheless, on the basis of the same approach, the result of factorized term is $(4-3)\tilde{x} = (1, 2, 3)$ meaning that FSIA-based approaches do not hold the factorization property. Nonetheless, the RDM-FIA method holds the factorization property and according to such method we have, $H(4\tilde{x} - 3\tilde{x}) = 1 + \alpha + 2\beta(1-\alpha) = H(\tilde{x})$ which reads $4\tilde{x} - 3\tilde{x} = \tilde{x}$. It should be noted that in the problem of tracking control of fuzzy dynamical systems the coefficients and factorized terms are fuzzy matrices with entries of fuzzy numbers that are much more complicated than the given simple example. Eventually, the drawbacks of FSIA-based approaches do not make it possible to solve the problem investigated in this work.

5 Simulation

Plane landing control

In this section, the purpose is to control the landing of the aircraft for landing. The jet is shown in Fig. 4. The direction x along the velocity vector of the plane, y and z are following the figure. The α is angle of attack, θ is the angle of the peak, and γ the angle of the flight path.

For simplicity, the following hypotheses are considered.

- Horizontal motion is avoided, and motion is considered only on the x-y plane.
- The speed of the plane during landing is considered to be almost constant. Due to the wind during landing, it is impossible to control the speed in an exact value, and a fuzzy number represents this approximate value.
- The flight path angle γ is considered small so that $\cos\gamma = 1$ and $\sin\gamma = \gamma$ when the angle is measured in radians.
- The longitudinal motion of the aircraft is entirely controlled by the deviation of the angle of the lift bulb

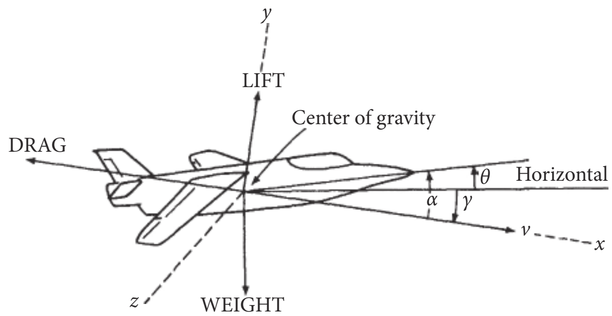


Fig. 4 Axes and angles of the plane.

$\delta_e(t)$ shown in Fig. 5.

● The dynamic equations governing the aircraft are expressed by a set of linear equations linear around the equilibrium flight conditions.

Because we want to easily have information about the system states needed to generate the control signal, the flight level altitude from the runway h, the altitude change velocity $\dot{h}$, the peak angle θ, and the peak angle change velocity $\dot{\theta}$ are selected as mode variables. If defined $X_4 \overset{\Delta}{=} \dot{\theta}, X_3 \overset{\Delta}{=} \theta, X_2 \overset{\Delta}{=} \dot{h}, X_1 \overset{\Delta}{=} h$ and $U \overset{\Delta}{=} \delta_e(t)$, the dynamic equations of the plane motion around the flight equilibrium conditions are Eq. (37).

$$\begin{gathered}
\begin{bmatrix} {}^{\mathrm{gr}}D\tilde{x}_1(t) \\ {}^{\mathrm{gr}}D\tilde{x}_2(t) \\ {}^{\mathrm{gr}}D\tilde{x}_3(t) \\ {}^{\mathrm{gr}}D\tilde{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{b}_4 \end{bmatrix} \tilde{U}(t), \\
\tilde{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \tilde{x}_3(t) \\ \tilde{x}_4(t) \end{bmatrix}
\end{gathered} \tag{37}$$

The coefficients $\tilde{a}_{ij}$ and $\tilde{b}_4$ related are the parameters of the plane, which are as follows in Eq. (38).

$$\begin{gathered}
\tilde{a}_{22} = -\frac{1}{\tilde{T}_s}, \quad \tilde{a}_{23} = \frac{\tilde{V}}{\tilde{T}_s}, \\
\tilde{a}_{42} = \frac{1}{\tilde{V}\tilde{T}_s^2} - \frac{2\xi\omega_s}{\tilde{V}\tilde{T}_s} + \frac{{\omega_s}^2}{\tilde{V}}, \quad \tilde{a}_{43} = \frac{2\xi\omega_s}{\tilde{T}_s} - {\omega_s}^2 + \frac{1}{\tilde{T}_s^2}, \quad \tilde{a}_{44} = \frac{1}{\tilde{T}_s} - 2\xi\omega_s, \\
\tilde{b}_4 = {\omega_s}^2 K_s \tilde{T}_s
\end{gathered} \tag{38}$$

In the above coefficients, the parameters K_s, $\tilde{T}_s$, ω_s, and ξ are as follows:

K_s :Fixed interest

$\tilde{T}_s$: Time constant

ω_s : Immortal natural frequency

ξ : Damping coefficient

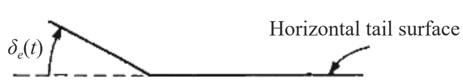


Fig. 5 Lifting angle deflection.

$\tilde{V}$: Speed

The numerical values considered for these parameters are

$$\begin{aligned}
&K_s = -0.95\ \text{s}^{-1},\\
&\tilde{T} = \widetilde{2.5}\ \text{s} = (2.2, 2.5, 2.7),\\
&\omega_s = 1\ \text{rad/s},\\
&\xi = 0.5,\\
&\tilde{V} = \widetilde{255}\ \text{ft/s} = (250, 255, 260)
\end{aligned} \tag{39}$$

The plane's speed is approximately $\tilde{V} = \widetilde{255}$ ft/s due to the wind. The altitude of the plane before landing is approx $\tilde{h} = \widetilde{110}(\text{ft}) = (105, 110, 115)$. The aircraft is unstable in the open-loop mode without control.

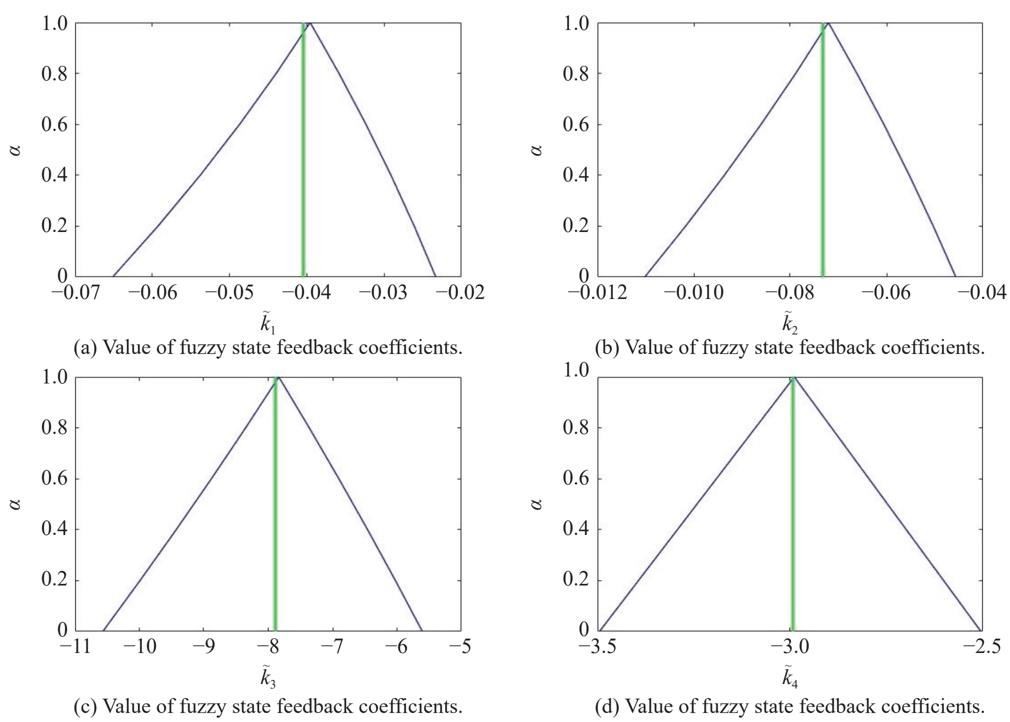
The article[42] discusses the design of state feedback for this issue, and this article examines the design of the state observer and the use of estimated modes for use in state feedback.

The goal is to design a controller with feedback coefficients $\tilde{\boldsymbol{k}} = [\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4]$ that converts system eigenvalues to $\tilde{\lambda}_1 = (-1.1, -1, -0.9)$, $\tilde{\lambda}_2 = (-1.2, -1.1, -1.05)$, $\tilde{\lambda}_3 = (-3, -2.5, -2)$, and $\tilde{\lambda}_4 = (-4, -3.5, -3)$. According to the article[42], the values of fuzzy state feedback coefficients are given in Fig. 6.

The membership function of the value of the determinants of the Eq. (19) given in Fig. 7. It is fully rated and therefore, an observable granular system.

The fuzzy eigenvalues of the observer are considered $\tilde{\lambda}_1 = (-0.5, -0.3, -0.1)$, $\tilde{\lambda}_2 = (-1, -0.8, -0.5)$, $\tilde{\lambda}_3 = (-1.8, -1.5, -1.3)$, and $\tilde{\lambda}_4 = (-2.5, -2.2, -2)$, and the gained matrix of the fuzzy observer $\tilde{\boldsymbol{l}} = [\tilde{l}_1, \tilde{l}_2, \tilde{l}_3, \tilde{l}_4]$ is obtained according to Fig. 8. The obtained non-fuzzy value is also shown in Fig. 8.

Figures 9 and 10 show the fuzzy output path $\tilde{h}(t)$ and its difference $\hat{\tilde{h}}(t)$. Figure 11 shows the fuzzy



(a) Value of fuzzy state feedback coefficients. (b) Value of fuzzy state feedback coefficients. (c) Value of fuzzy state feedback coefficients. (d) Value of fuzzy state feedback coefficients.

Fig. 6 Fuzzy state feedback gain. Green lines are non-fuzzy values.

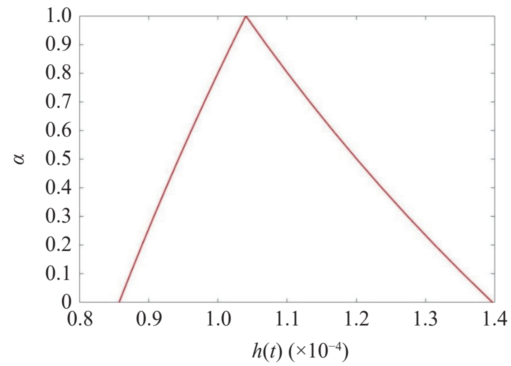


Fig. 7 Determinants of the granular observable matrix.

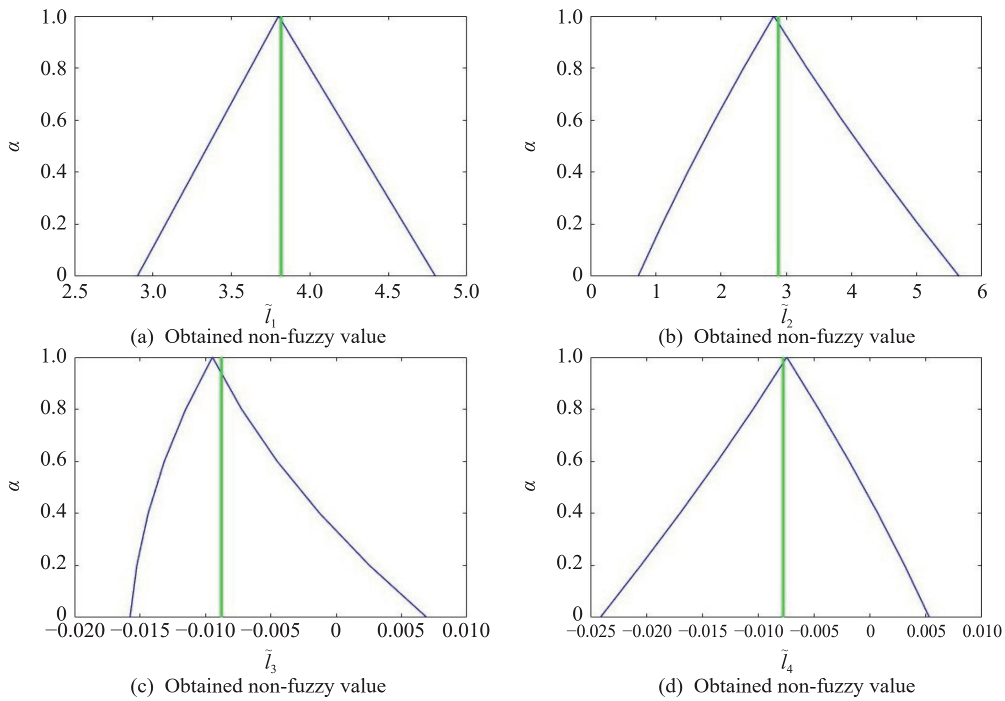


Fig. 8 Fuzzy observer gain. Green lines are non-fuzzy values.

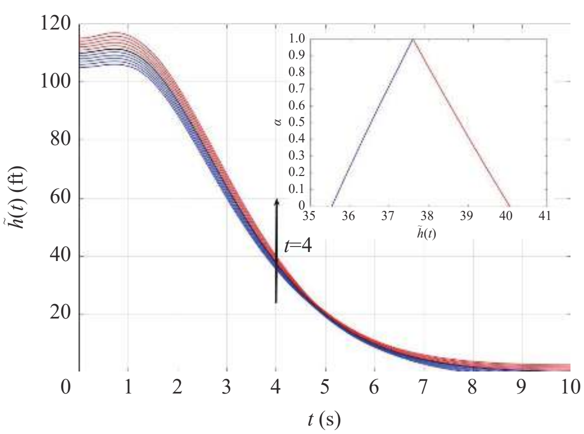


Fig. 9 Fuzzy output trajectory $\hat{\tilde{h}}(t)$.

motions of the jet plane considering the fuzzy conditions. As can be seen, the controller and the observer could control the landing of the aircraft well.

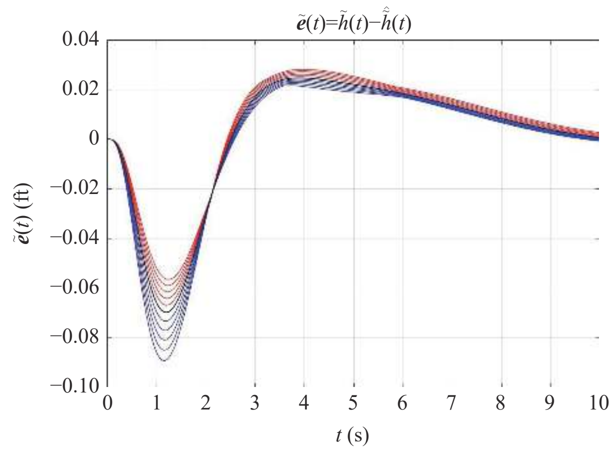


Fig. 10 **The difference between two fuzzy variables $\tilde{h}(t)$ and $\hat{\tilde{h}}(t)$.**

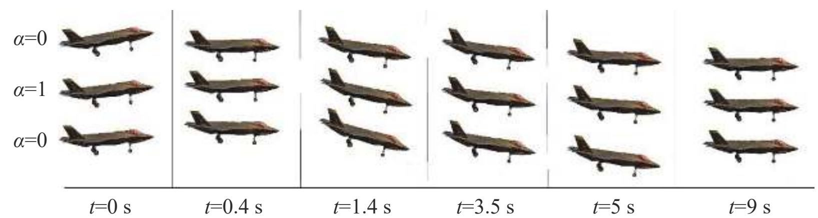


Fig. 11 **Expected movements of aircraft in the presence of the fuzzy controller and the observer.**

It is further assumed that due to the sudden wind, the altitude of the aircraft is disturbed between 3 and 3.5 seconds, which this disturbance $\boldsymbol{d}(t) = [d(t) \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$ is shown in Fig. 12. Furthermore, again the same controller and observer of the previous step are applied to the system.

Figure 13 and Fig. 14 show the fuzzy trajectory of output $\tilde{h}(t)$ and its difference with $\hat{\tilde{h}}(t)$ and in Fig. 15 as an example of the difference error between $\tilde{x}_3(t)$ and $\hat{\tilde{x}}_3(t)$. As can be seen, the control system has been able to reject the disturbance in the system well.

6 Discussion

As was already mentioned, the proposed method is based on fuzzy mathematics which makes it possible to consider uncertainty in the system mathematical model parameters as fuzzy sets. Such a viewpoint enables

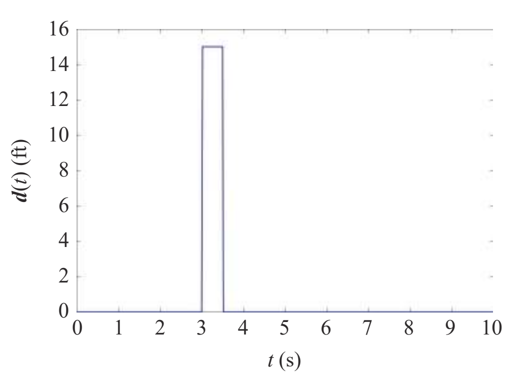


Fig. 12 **Disturbance $\boldsymbol{d}(t) = [d(t) \; 0 \; 0 \; 0]^{\mathrm{T}}$.**

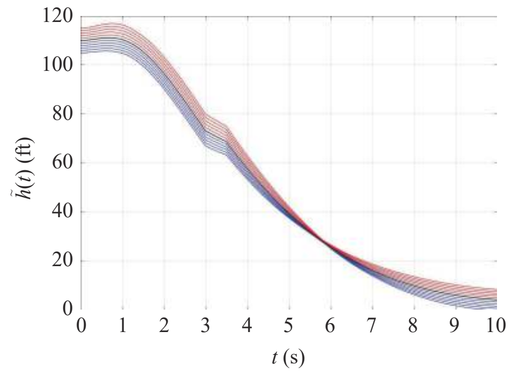


Fig. 13 Fuzzy output trajectory $\tilde{h}(t)$ in the presence of disturbance.

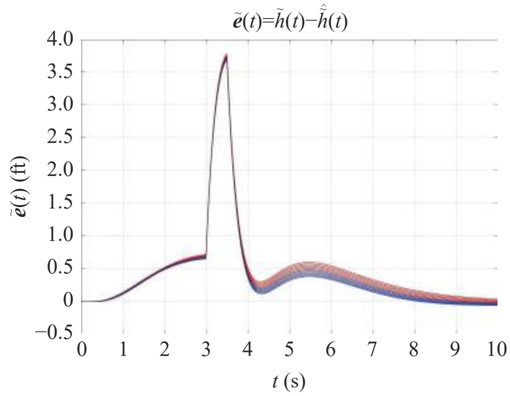


Fig. 14 Difference between $\tilde{h}(t)$ and $\hat{\tilde{h}}(t)$.

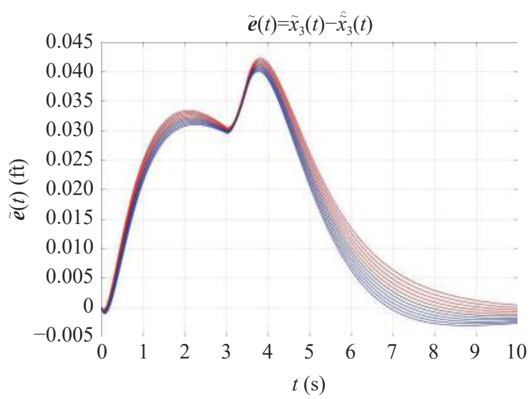


Fig. 15 Difference between $\tilde{x}_3(t)$ and $\hat{\tilde{x}}_3(t)$.

us to obtain state and control trajectories with some information about the possibility distribution of the states and control. According to Ref. [44], such a distribution can be interpreted in many ways among which is the degree of possibility and easiness. As an instant, the fuzzy output trajectory illustrated in Fig. 9 shows the degree of easiness or possibility that the plane may pass the path with different altitude. As a case in point, at t=4, the possibility that the plane altitude is 37.6 ft. However, with 50% possibility, the plane altitude is 39 ft.

In addition, the disturbance rejection ability is one of the major features that a controller should possess. Depending on the range of disturbance types, the ability of controllers differs from each other. As a matter of fact, the more the range of disturbance types, the more complicated the controller structure, and the more able the controller to reject the disturbance. In the proposed control structure, the disturbance rejection feature includes those which are stationary (with a low fluctuation). Therefore, as long as the disturbance type is stationary then the controller can reject it desirably. However, if the disturbance varies with high frequency and amplitude, then the controller may fail to achieve the rejection suitably.

Moreover, it should be noted that based on the granular calculus, the fuzzy differential equations system, in this work, has been converted to its corresponding granular differential equations system which is a system of crisp differential equations with respect to each degree of membership and the value of the RDM variable. Therefore, as explained in Ref. [19], the existence and uniqueness of the solution of fuzzy differential equations is the same as that of its corresponding crisp differential equations.

Additionally, it should be noted that in comparison with Refs. [42, 45], the proposed fuzzy control structure illustrated in Fig. 3 is the more general case. Specifically, the fuzzy control system presented in Ref. [42] is unable to reject the disturbance and lacks fuzzy observer, moreover, the controller designed in Ref. [45] lacks the fuzzy dynamic compensator and it is not able to reject the disturbance. Nevertheless, the proposed fuzzy controller in this work possesses disturbance rejection feature, fuzzy dynamic compensator provided by the granular fuzzy integral, and it has been equipped with the fuzzy observer in order to estimate uncertain system sates. Moreover, in this work the proposed fuzzy controller has been designed through the branch of fuzzy mathematics which completely differs from the traditional fuzzy if-then rules method, e.g., Ref. [46].

7 Conclusion

In this paper, the problem of fuzzy tracking control for fuzzy linear dynamic systems is investigated based on the concepts of granular derivatives and the mathematics of fuzzy intervals of relative distance measure. These systems consider uncertainty as fuzzy numbers, which represent the knowledge of an expert. Using the theorem stated in Ref. [42], it is demonstrated that the system output, which has uncertain parameters, can track the reference input. The fuzzy control input takes the form of system fuzzy state feedback. As it is not always possible to measure system states, a fuzzy observer was designed to estimate the system states in this paper, and the necessary condition for designing a fuzzy observer, which is the same as granular observation, was presented. The fuzzy numbers in the fuzzy differential equations are obtained from the expert's knowledge, and the more information and knowledge the expert has, the less uncertainty there is in the fuzzy numbers and, of course, the less uncertainty in the feedback gain of the cases, the more accurate the results are. This paper uses the fuzzy interval mathematics approach, the relative distance measure, to address the problems in SGH and gH methods, such as the absence of derivative solutions and the phenomenon of unnatural behavior in modeling. Finally, an example is used to investigate the effectiveness of a fuzzy controller with a designed fuzzy observer.

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